Master ICFP | Quantum Physics

Atoms and photons

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Duration three hours. Lectures notes, slides handouts (even on tablets), exercise class notes and manuscript documents are authorized. Books, mobile phones, computers are not. Numerical estimates are expected only at the order of magnitude level, so that calculations can be performed without a handheld calculator.

Preliminary

In 1999, Lene Hau and her group reported the observation of very slow light propagation in a cold sodium atoms medium (Nature, **397**, 594). By using Electromagnetically Induced Transparency, they were able to demonstrate the propagation of light pulses with velocities as low as 17 m/s, corresponding obviously to a very large index of refraction. Rather counterintuitively, this large refraction does not come at the expense of a large absorption.

This problem uses the formalism of the Maxwell-Bloch equations to describe this experiment. The first introductory part recalls standard results about absorption and dispersion in a two-level atom sample. The bulk of the problem then focuses on the experimental situation of a three-level Lambda structure addressed by two laser fields, a resonant intense pump field and a weak probe field, on which the slow propagation is observed.

I. Introduction

We consider the propagation of light in a sample of N two-level atoms per unit volume, with a ground state $|g\rangle$ (zero energy) and an excited state $|e\rangle$, with energy $\hbar\omega_e$. The $|g\rangle \to |e\rangle$ transition has a dipole matrix element d (considered, without loss of generality, to be real). The excited level $|e\rangle$ decays towards $|g\rangle$ with a spontaneous emission rate $\Gamma = \omega^3 d^2/3\pi\epsilon_0\hbar c^3$. The environment is at zero temperature. We will not consider here any other source of decoherence for the atomic system. The atomic state is described by the density matrix, which reads in the $\{|e\rangle, |g\rangle\}$ basis:

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix}$$

It is driven by a classical probe laser field $E_s \exp(-i\omega t) + c.c.$ (Note that s here stands for 'sonde'). We will assume for the sake of simplicity that E_s is real and introduce the probe Rabi frequency $\Omega_s = dE_s/\hbar$. The probe detuning is $\Delta = \omega_e - \omega$ (note that the sign convention for the detuning is not the same here – and in the lecture notes – as in the TD). The light propagates in the sample over a total length L. The light wavelength at resonance with the atomic transition is λ.

1. Recall, from the lecture notes, the optical Bloch equations for $\tilde\rho_{ee}$ and $\tilde\rho_{eg}$ in interaction representation with respect to the probe laser frequency.

2. We are interested in the weak probe regime, with an intensity far below the saturation. We thus set $\tilde\rho = \tilde\rho^{(0)} + \Omega_s \tilde\rho^{(1)}$ (note that, with this definition, $\tilde\rho^{(1)}$ and $\tilde\rho^{(0)}$ do not have the same dimension). The atoms are initially in state $|g\rangle$. Give $\tilde\rho^{(0)}$. Write the equation of evolution of $\tilde\rho^{(1)}_{eg}$ at the relevant order.

3. Give the steady state solution of this equation.

4. Give the density of polarization of the atomic medium, $\mathbf{P}$. Deduce that the electric susceptibility, $= \ ' + i \ ''$, in the limit of a dilute atomic medium, is $= {}_0 2\Gamma{=}(2 \quad i\Gamma)$, where ${}_0 =$ $N d^2{=}(\hbar \, {}_0 \Gamma)$. Give the expression of ${}_0$ as a function of N and . Give $'$ and $''$. Under which condition is the dilute medium approximation valid?

5. Give in the same limit the index of refraction $n = n' + in''$. Plot qualitatively the variation of n' and n'' with .

6. Compute at resonance the characteristic length L_0 of the exponential decay of the light intensity. Give a simple expression of L_0^{-1} in terms of the resonant absorption cross section $= 3 \ ^2{=}2$. Interpretation.

7. The optical density of the medium is defined as $OD = L{=}L_0$. What is its physical interpretation? Give an order of magnitude of L_0 and of the OD for Hau's experiment, based on the paper data: $L = 229$ m; $= 589$ nm; $N = 3{:}10^{12}$ cm^{-3}; $\Gamma = 6{:}1\,10^7$ rad/s. Conclusions?

8. Give also an order of magnitude estimate of the maximum real part of the index of refraction. Are our hypotheses correct?

II. Three-level OBEs

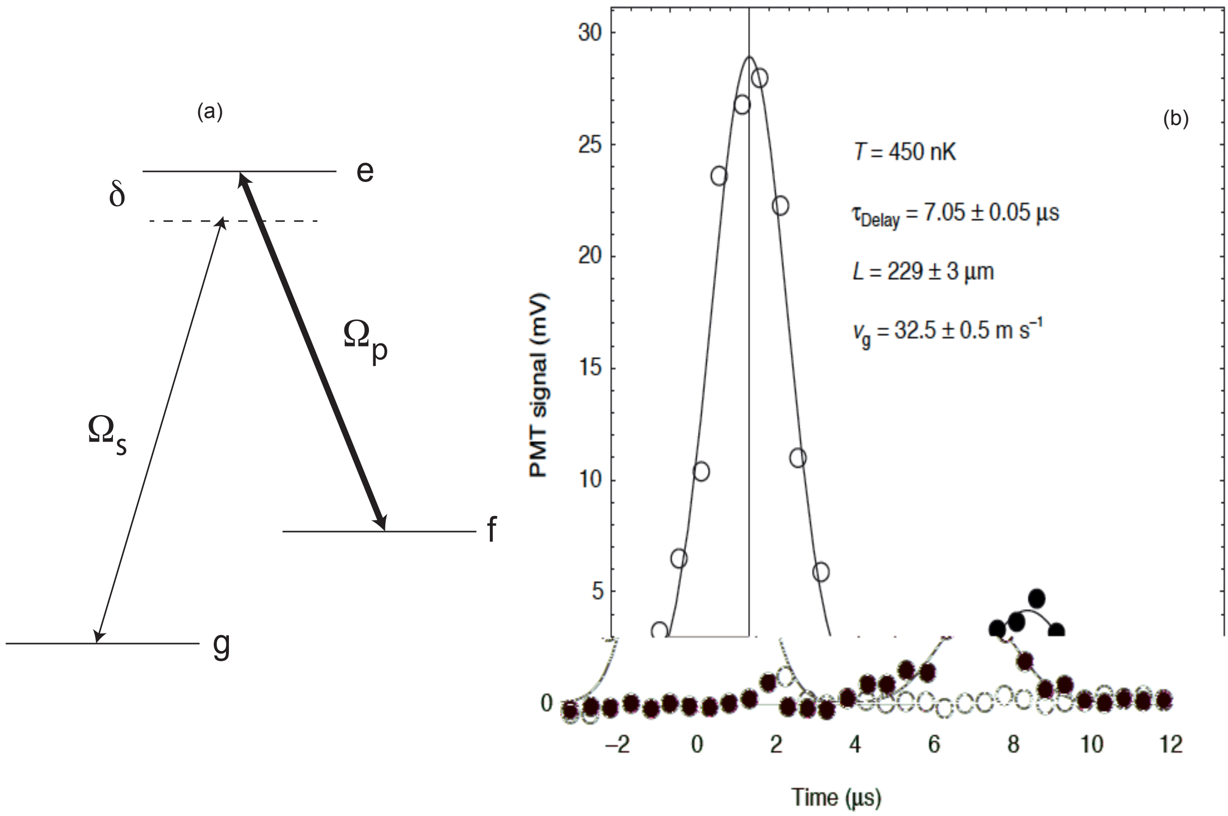

Figure 1: (a) Simplified level scheme. (b) experimental light propagation signal.

We now switch to the actual three-level configuration used in the paper (see figure 1). In addition to jei and jgi (ground state with zero energy), we consider level jfi, with energy $\hbar !_f$. In the experiment, jgi and jfi are two hyperfine levels of the sodium atom and can both be considered to have an infinite lifetime. The $|f\rangle \;\omega\; |e\rangle$ transition is driven by an 'intense' pump laser (frequency ω_p), with a Rabi frequency Ω_p (assumed to be real). The pump laser is assumed throughout this problem to be at exact resonance: $\omega_p = \omega_{ef} = \omega_e \quad \omega_f$. The $|g\rangle \;\omega\; |e\rangle$ transition is driven by a weak 'probe' laser with the Rabi frequency Ω_s and the frequency $\omega_s = \omega_e \quad$. The atom is initially in $|g\rangle$.

In the first part of this section, we consider the Hamiltonian equations of motion of the atomic density matrix. We then add the relaxation terms. Finally, we give a perturbative solution for the steady state. Along the calculation, we will focus only on the density matrix elements that are relevant for our discussion.

1. Show that, keeping only resonant and relevant terms, the full atomic Hamiltonian H can be put under the form:

$$H/\hbar = \omega_e|e\rangle\langle e| + \omega_f|f\rangle\langle f| \quad \frac{\Omega_s}{2}\left[e^{-i\omega_s t}|e\rangle\langle g| + \text{h.c.}\right] \quad \frac{\Omega_p}{2}\left[e^{-i\omega_p t}|e\rangle\langle f| + \text{h.c.}\right]$$

   Cast this in matrix form in the $\{|e\rangle;|f\rangle;|g\rangle\}$ basis. Note that we will not use interaction representation before question 6.

2. Deduce the equations for the Hamiltonian evolution of $\;_{eg}$ and $\;_{fg}$. Is this set of equations closed?

3. We now proceed to include relaxation. What are a priori all the possible quantum jump operators compatible with our hypotheses?

4. In order to capture the essential physics without algebraic complications, we assume that $|e\rangle$ only decays towards $|g\rangle$ with the rate Γ. By a simple generalization of the two-level case, give the evolution equations for $\;_{eg}$ and $\;_{fg}$.

5. As in the introductory part, we look for a perturbative solution of the Bloch equations for low probe powers, i.e. for a low value of Ω_s. We thus set: $\;= \;^{(0)} + \Omega_s \;^{(1)}$ (same note as in question I2). Give $\;^{(0)}$. Write the equations of motion of the relevant elements of $\;^{(1)}$. Show that the resulting set of equations is closed

6. These equations contain an explicit time dependence. It can be removed by switching to an interaction representation. We thus set $\tilde{\;}^{(1)}_{eg} = \;^{(1)}_{eg}e^{i\omega_1 t}$ and $\tilde{\;}^{(1)}_{fg} = \;^{(1)}_{fg}e^{i\omega_2 t}$. Write the equations of motion of these new quantities. Which values of ω_1 and ω_2 make all coefficients in the evolution equations time-independent?

7. Show finally that the relevant equations of motion are the closed set:

$$\begin{aligned}\frac{d\tilde{\;}^{(1)}_{eg}}{dt} &= (i \quad + \Gamma/2)\,\tilde{\;}^{(1)}_{eg} + \frac{i}{2} + i\frac{\Omega_p}{2}\tilde{\;}^{(1)}_{fg}\\ \frac{d\tilde{\;}^{(1)}_{fg}}{dt} &= i \quad \tilde{\;}^{(1)}_{fg} + i\frac{\Omega_p}{2}\tilde{\;}^{(1)}_{eg}\end{aligned}$$

8. Give the stationary state solution of these two equations. Discussion.

III. Slow light propagation

We examine now the light propagation in the medium made up of the atoms described in the previous section.

1. By arguments similar to those of the introductory part, give the electric susceptibility $\quad$ of the medium for the probe field, as well as its real and imaginary parts. What happens for $\;= 0$?

2. Give the inverse of the absorption length in these conditions, L_1^{-1}, as a function of L_0, Γ, , and Ω_p.

3. Recall, from the lecture notes, the aspect of the function $L_1^{-1}(\ )$ in the case of a large pump intensity ($\Omega_p \quad \Gamma$). Give a physical interpretation.

4. Recall, from the lecture notes, the shape of $L_1^{-1}(\ )$ in the other limit, $\Omega_p \quad \Gamma$. Is the name 'Electromagnetically Induced Transparency' reasonable for what happens?

5. Give a physical interpretation of the total absence of absorption at resonance.

6. Evaluate the full width at half maximum, Δ, of the transparency window in the $\Omega_p \quad \Gamma$ regime. We stay in this regime from now on. Discuss the $\Omega_p\ !\ 0$ limit.

7. Give an approximate expression of L_1^{-1} as a function of   near the center of the transparency window. We set rather arbitrarily as the transparence criterion that L_1^{-1} should be less than L^{-1}. Compute the allowed range of   values as a function of Ω_p, Γ and OD, the optical density for the two-level situation (no pump field).

8. Give an approximate expression of the real part of the index of refraction n' near the centre of the transparency window.

9. In a medium with an index of refraction $n(\omega)$, the group velocity is $v_g = c/(n + \omega\, dn/d\omega)$. Compute the group velocity in the centre of the transmission window.

10. Show that the group velocity is close to the product of the two-level absorption length L_0 by the width Δ of the transparency window. Compare that with the speed of light and conclude.

11. Give an order of magnitude estimate of v_g in the conditions of Hau's experiment, with $\Omega_p = 3.6\,10^7$ rad/s. Compare this order of magnitude with the observation of figure 1. Are all our hypothesis fully satisfied in this regime?

12. Does the slow light phenomenon apply to arbitrary input wave packets? What are the conditions to observe it properly? What happens when they are not satisfied?