Quantum computing
Lecture 1

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Introduction

Topic of the course: realization of experimental platforms for quantum computation or quantum simulation

- Mastering single quantum objects = ‘qubits’ and use entanglement: first pioneered with photons (quantum optics), then atoms and ions, and finally with solid state devices: superconducting transmon qubits, spin qubits...

- Realize single qubit gates (rotations on the Bloch sphere) and two-qubit gates (preparing entangled states)

- A common formalism valid for all platforms: quantum computation (see Perola Milman’s lectures)
Introduction

Topic of the course: realization of experimental platforms for quantum computing or quantum simulation

- A common problem to face: decoherence. Challenge: increase the number of operations within the decoherence time.
- Possible trade-off: Development of error correction schemes with several physical qubits to encode a single logical qubit.
- No universal quantum computer available yet! Rather quantum simulators with various experimental advantages and limitations. The best choice depends on the problem to address.
Criteria for a quantum computer

David DiVincenzo proposed five criteria that a quantum computer should fulfill:

- A **scalable** physical system with well characterised qubits.
- The ability to **initialize** the state of the qubits to a simple fiducial state.
- Long relevant **decoherence times**.
- A universal set of **quantum gates**.
- A qubit-specific **measurement** capability.

The criteria are less stringent on quantum simulators (especially on scalability and universality).
The platforms

Experimental platforms studied in this course (tentative list)

- Trapped ions
- Transmon (superconducting) qubits
- NMR qubits
- Electron spins in quantum dots
- Electron spins in silicium
- Photons
- Rydberg atoms
- Ultracold atoms in optical lattices
Outline of this course

Lecture 1: Quantum computation with trapped ions

1. Trapping charged particles
2. Laser manipulation of trapped ions
3. Coupling ions via phonons
4. Example of quantum algorithms performed with trapped ions
Outline of this course

Lecture 2: Quantum computation with transmon qubits

1. The superconducting transmon qubit
2. Manipulation of a single qubit with microwave fields
3. Two qubit gates
4. Example of quantum algorithms performed with transmon qubits
Outline of this course

Lecture 3: Other quantum computation platforms

1. NMR in molecules
2. Spins in quantum dots
3. Silicium qubits
4. Photons
Outline of this course

Lecture 4: Quantum simulation with neutral atoms

1. Rydberg atoms in micro-traps
2. Quantum gases in optical lattices
3. Quantum gases in the bulk
Bibliography

- lecture notes by John Preskill
  http://www.theory.caltech.edu/people/preskill/ph229/


- C. Champenois, *Trapping and cooling of ions*, Les Houches lecture notes
  https://cel.archives-ouvertes.fr/cel-00334152


Lecture 1: Quantum computation with trapped ions
Outline

Introduction

Trapping charged particles
  - Penning trap
  - Paul trap
    Quantized motion in the trap

Laser manipulation of trapped ions
  - Reminder on atom-light interaction
  - Laser cooling to the ground state
  - Qubit manipulation and detection

Coupling ions via phonons
  - Phonons as control qubit
  - Center of mass mode of an ion chain
  - Phonon-mediated two-qubit gates

Example of a quantum algorithm with trapped ions
Trapping charged particles
Characteristics of a ‘good’ ion trap

Objective: ability to manipulate the internal degree of freedom of single ions (qubit) and couple ions for two-qubit gates to satisfy the DiVincenzo criteria.

- trap should not affect the qubit transition ⇒ confinement relying on the charge itself
- absolute control on the external degree of freedom ⇒ tight trap and cooling to the ground state
- addressability of individual ions ⇒ separation larger than the resolution (esp. for optical transitions)
- avoid decoherence ⇒ controlled environment (ultra-high vacuum, low noise, etc.)
Trapping charged particles with electric fields

A static electric field cannot confine a charged particle \((\nabla \cdot E = 0\) or \(\Delta U = 0\)). We can only get two directions trapped and the other antitrapped, with a potential between two conductors of the form:

\[
U(x, y, z) = \frac{1}{2} M \omega^2 \left( \frac{x^2 + y^2}{2} \right) - z^2.
\]

To stabilize the particle in 3D, there are two options:

- Add a magnetic field: **Penning trap**
- Make the electric field time-dependent: **Paul trap**
The Penning trap

Electric confinement along $z$

$$U(x, y, z) = \frac{1}{2} M \omega^2 \left( -\frac{x^2 + y^2}{2} + z^2 \right).$$

Magnetic confinement in the plane

Magnetic field $B$ along $z$ with a cyclotron frequency $\omega_c = qB/M$ much larger than $\omega$. The in-plane motion is decomposed into a fast rotation at $\sim \omega_c$ and a slow oscillation at the magnetron frequency $\omega_m$

$$\omega_m \sim \frac{\omega^2}{2\omega_c} \ll \omega \ll \omega_c$$
Trapping charged particles

Penning trap

Trajectory in a Penning trap

Typical trajectory

Main usages

- Trapping single particles for metrology (measurement of mass ratio, measurement of \( g - 2 \) of a single electron...)
- Study ion crystals

Drawback for quantum computation:

Lower trapping frequency of the magnetron motion (typ. 10-100 kHz).
The Paul trap

Key idea: alternate the voltage fast enough \((\Omega \gg \omega)\) such that the particle remains trapped in the time-averaged potential.

\[
U(x, y, z, t) = \frac{1}{2} M \omega^2 \left(-\frac{x^2 + y^2}{2} + z^2\right) \cos(\Omega t).
\]

Decompose the motion into fast + slow: \(\mathbf{r} = \mathbf{r}_f + \mathbf{r}_s\) with \(\mathbf{r}_f \ll \mathbf{r}_s\). On the fast scale \(\Omega\), \(\mathbf{r}_s\) is fixed:

\[
\ddot{\mathbf{r}}_f = -\frac{\omega^2}{2} \left(\frac{-\rho_s}{2z_s}\right) \cos(\Omega t) \Rightarrow \mathbf{r}_f = \frac{\omega^2}{2\Omega^2} \left(\frac{-\rho_s}{2z_s}\right) \cos(\Omega t).
\]

Now the slow motion obeys the time-averaged equation:

\[
\ddot{\mathbf{r}}_s = -\frac{\omega^2}{2} \left\langle\left(\frac{-\rho_f \cos(\Omega t)}{2z_f \cos(\Omega t)}\right)\right\rangle = -\frac{\omega^4}{8\Omega^2} \left(\frac{\rho_s}{2z_s}\right).
\]
The Paul trap: average potential

Final potential for the slow motion:

\[ V(x_s, y_s, z_s) = \frac{M \omega^4}{16 \Omega^2} \left( x_s^2 + y_s^2 + 4z_s^2 \right). \]

⇒ Harmonic trap with frequencies \( \omega_{\perp} = \frac{\omega^2}{2\sqrt{2}\Omega} \) and \( \omega_z = 2\omega_{\perp} \).

Remarks:

- \( \Omega \) is in the rf domain, Paul traps are also called rf traps.
- \( \omega_z, \perp \ll \omega \ll \Omega \) so it’s important to start with a very steep electric potential (easier for small-scale traps).
- \( V(r_s) = \frac{M v_f^2}{2} \): the trap relies on the kinetic energy of the micromotion. This motion is hence more important for a particle far from the center. It vanishes at the center where the fields cancel.
- The stability of the trap is valid only under some assumptions. See e.g. the Mathieu stability diagramme in the course of C. Champenois.
The Paul trap: 3D vs linear

The Paul trap can be implemented in a linear quadrupole version, ideal to confine a linear ion chain (a static field is added to trap the z motion):

Linear Paul traps are ideal to confine many ions while limiting the micromotion (ion chain at the center $x = y = 0$). They can be made smaller ⇒ reach higher frequencies ($\sim 10$ MHz). They are used in optical metrology of time or quantum optics and quantum computing.
Linear Paul trap in practice

End caps with positive voltage $U_{\text{end}} > 0$ to confine the weak ($x$) axis.

Innsbruck linear Paul trap (2000)
Linear Paul trap in practice

The end caps can be placed at the end of the ground rods to improve optical access.

Figure from Ludlow et al., RMP 87, 637 (2015)
Going small: Surface ion traps

Surface ion trap design of the Boulder group (IonQ company, Chris Monroe). 80 atomic $^{171}Yb^+$ ions.

Figure from Monroe et al., Science **364**, 440 (03 May 2019)
Trapped ion: classical vs quantum motion

Consider a harmonic trap with frequency $\omega_0$ along a direction $x$. Two regimes depending on the ion(s) temperature:

- $k_B T \gg \hbar \omega_0$: Many trap levels are populated, the motion can be described classically with position $r$ and velocity $v$.
- $k_B T \ll \hbar \omega_0$: The atomic motion is quantum mechanical.

Hamiltonian of the harmonic oscillator:

$$H_{ext} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2$$

with $[\hat{x}, \hat{p}] = i\hbar$.

Ground state has a position rms width $x_0$ and a momentum rms width $p_0$

$$x_0 = \sqrt{\frac{\hbar}{2m\omega_0}} \quad p_0 = \sqrt{\frac{\hbar m \omega_0}{2}}$$

such that $x_0 p_0 = \hbar/2$. 
Trapped ion: classical vs quantum motion
Trapped ion: classical vs quantum motion

Consider a harmonic trap with frequency $\omega_0$ along a direction $x$. Two regimes depending on the ion(s) temperature:

- $k_B T \gg \hbar \omega_0$: Many trap levels are populated, the motion can be described **classically** with position $r$ and velocity $v$.
- $k_B T \ll \hbar \omega_0$: The atomic motion is **quantum mechanical**.

Hamiltonian of the harmonic oscillator:

$$H_{\text{ext}} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2 = \hbar \omega_0 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

Eigenstates $|n\rangle_{\text{ph}}$ with energies $E_n = (n + \frac{1}{2}) \hbar \omega_0$.

$\hat{a}$ and $\hat{a}^\dagger$, such that $[\hat{a}, \hat{a}^\dagger] = 1$, are **annihilation** and **creation** operators of vibrations in the trap = **phonons**.

$$\hat{a} |n\rangle_{\text{ph}} = \sqrt{n} |n-1\rangle_{\text{ph}} \quad \hat{a}^\dagger |n\rangle_{\text{ph}} = \sqrt{n+1} |n+1\rangle_{\text{ph}}$$

$$\hat{x} = x_0 \left( \hat{a} + \hat{a}^\dagger \right) \quad \text{with} \quad x_0 = \sqrt{\frac{\hbar}{2m \omega_0}}.$$
Laser manipulation of trapped ions
Laser manipulation of trapped ions

Laser manipulation is essential to quantum computation with trapped ions. It serves to

- **Cool** the external motion (the phonons) to the trap ground state (two-step cooling: Doppler + resolved sideband cooling)
- Prepare the initial (internal) state of the qubit
- Perform single- or two-qubit gates
- Read the qubit state
Ions used for quantum computation as well as clocks are typically alkaline earth atoms ($\text{Be}^+$, $\text{Mg}^+$, $\text{Ca}^+$, $\text{Sr}^+$) or ytterbium $\text{Yb}^+$. They present a **short lived excited state** $|e\rangle$ of width $\Gamma$ (ex: $P_{3/2}$) strongly coupled by an electric dipole transition to the **ground state** $|g\rangle$ (ex: $S_{1/2}$) which can be used for laser Doppler cooling + weaker transitions to **long-lived states** (ex: coupling to $D_{5/2}$ state $|d\rangle$) useful for resolved sideband cooling or gate operations. Figure from Bruzewicz et al.
Focus on a two-level system

- Interaction between an atom or ion and nearly resonant light with a particular transition $g \to e$.
- Focus on a two-level system with ground state $|g\rangle$ and excited state $|e\rangle$: $H_0 = \hbar \omega_{eg}/2 [|e\rangle \langle e| - |g\rangle \langle g|]$. 
- All other levels can be ignored.
- This two-level system maps onto a $1/2$-spin:

\[
H_0 = \frac{\hbar \omega_{eg}}{2} \sigma_z \quad \text{with} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

- When the atomic motion is relevant (e.g. for cooling) the ion hamiltonian reads

\[
H_{\text{ion}} = H_0 + \frac{\hat{p}^2}{2m} + V_{\text{trap}}(\hat{r})
\]

treated classically or quantum mechanically depending on temperature $T$. 

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Reminder on atom-light interaction (semi-classical)

Atom at position $r$ coupled to a monochromatic field at $\omega$

- Internal state Hamiltonian: $H_0 = \frac{\hbar \omega_{eg}}{2} \left[ |e\rangle \langle e| - |g\rangle \langle g| \right] = \frac{\hbar \omega_{eg}}{2} \sigma_z$.

- Interaction Hamiltonian in the dipolar approximation:

$$H = \left( \frac{\mathbf{p} - q \mathbf{A}(r)}{2m_e} \right) + \frac{q U(r)}{\text{int. with nucleus} \atop \text{nucleus}} + q V(r)$$

$$\mathbf{A}(r) \rightarrow \mathbf{A}(r) \mathbf{e}_r$$

$$V(r) \rightarrow V(r) + r \mathbf{E}$$

$$H = H_0 - \frac{q}{m_e} \mathbf{p} \cdot \mathbf{A}(r) + q \mathbf{e}_r \cdot \mathbf{E}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}(r)}{\partial t} - \mathbf{E}$$

$$\chi = -\mathbf{A}(r) \cdot \mathbf{e}_t$$

$$\chi \rightarrow \chi + \mathbf{E}$$

$$V \rightarrow -\frac{\partial \mathbf{E}}{\partial t}$$

$$\Rightarrow H_\chi = -\mathbf{E} \cdot \mathbf{E}$$

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- Internal state Hamiltonian: $H_0 = \frac{\hbar \omega_{eg}}{2} [|e\rangle \langle e| - |g\rangle \langle g|] = \frac{\hbar \omega_{eg}}{2} \sigma_z$.

- Interaction Hamiltonian in the dipolar approximation:

$$H_I = -\hat{D} \cdot E(\hat{r}, t)$$

- $\langle s| \hat{R}_e |s\rangle = 0$ in an atomic state $s$ of well-defined parity $\Rightarrow$ Dipole operator $\hat{D} = q \hat{R}_e$ is purely off-diagonal in the $\{|g\rangle, |e\rangle\}$ basis:

$$\hat{D} = d |e\rangle \langle g| + d^* |g\rangle \langle e| = d\sigma_+ + d^*\sigma_-$$

and cancels if $|e\rangle$ and $|g\rangle$ have the same parity.

- Simple case: real dipole $d$, linearly polarized plane wave:

$\hat{D} = d(\sigma_+ + \sigma_-)$

$E(r, t) = E_0 \cos(\omega t + \varphi - k \cdot r) = E_0 e_x \cos(\omega t + \varphi - k \cdot r)$. 
Atom-field coupling

Rabi frequency

Atom-field Hamiltonian:

\[
H_I = -d \cdot E_0 \cos(\omega t + \varphi - k \cdot \hat{r})(\sigma_+ + \sigma_-)
\]

\[
H_I = \frac{\hbar \Omega}{2} \left( e^{i\omega t + i\varphi - i k \cdot \hat{r}} + e^{-i\omega t - i\varphi + i k \cdot \hat{r}} \right) (\sigma_+ + \sigma_-)
\]

with definition of the Rabi frequency

\[
\Omega = -\frac{d \cdot E_0}{\hbar}
\]

In the case where both \( \Delta = \omega - \omega_{eg} \) and \( \Omega \) are much smaller than \( \omega \), Rotating wave approximation (RWA) applies and we can write (see later)

\[
H_I \simeq \frac{\hbar \Omega}{2} \left( e^{-i\omega t - i\varphi + i k \cdot \hat{r}} \sigma_+ + e^{i\omega t + i\varphi - i k \cdot \hat{r}} \sigma_- \right)
\]
Coherent coupling vs spontaneous decay

Coupling to the empty modes of the field

Fermi Golden rule applied between excited atom and empty modes of the field coupled by electric dipolar coupling, near the frequency $\omega \sim \omega_{eg}$

$$\Gamma \propto \left| \langle g | W | e \rangle \right|^2 \propto d^2$$

$$\Gamma = \frac{d^2 \omega_{eg}^3}{3\pi \hbar \varepsilon_0 c^3}$$

$\Gamma$: linewidth of the transition

Limit cases:

- $\Gamma \gg \Omega$: strong transition / short-lived state $|e\rangle$: evolution of the density matrix $\hat{\rho}$ to compute the light forces (OBE).

- $\Gamma \ll \Omega$: weak transition / long-lived state $|e\rangle$: hamiltonian evolution (Rabi oscillations) of quantum states $|\psi\rangle$ for quantum gates
Short-lived excited state and moving atom: Light forces

Classical atom motion

For a free two-level atom, classical external motion \((\mathbf{r}, \mathbf{v})\), interacting with a near resonant classical laser field

\[
E(\mathbf{r}, t) = E_0 u_x \cos(\omega t + \varphi - \mathbf{k} \cdot \mathbf{r})
\]

on can derive light forces from the dipolar interaction \(-\mathbf{D} \cdot \mathbf{E}(\mathbf{r}, t)\) and the optical Bloch equations (see e.g. CCT/DGO book).
Short-lived excited state and moving atom: Light forces

Classical atom motion

For a free two-level atom, classical external motion \((r, v)\), interacting with a near resonant classical laser field

\[
E(r, t) = \mathcal{E}_0 u_x \cos(\omega t + \varphi - k \cdot r)
\]

one can derive light forces from the dipolar interaction \(-\hat{D} \cdot E(r, t)\) and the optical Bloch equations (see e.g. CCT/DGO book). Radiation pressure force in the direction of the laser due to momentum conservation, computed at detuning \(\Delta = \omega - \omega_{eg} - k \cdot v\)

\[
F_{pr}(v) = \frac{\Gamma}{2} \frac{s_0}{1 + s_0 + 4\Delta^2/\Gamma^2} \hbar k
\]

\(\Gamma\): linewidth (inverse lifetime) of the excited state, large enough.

With a red detuned laser beam, i.e. \(\omega < \omega_{eg}\), the force is larger if \(v\) is opposite to the laser direction \(k\).
Laser cooling of the (classical) ion motion

(1) Doppler cooling

**Trap of frequency** $\omega_0$. Two successive steps.

(1) **Doppler cooling** on a broad transition $\Gamma \gg \omega_0$ ($\Gamma^{-1}$ lifetime of the electronic excited state) when the ion motion is still classical ($k_B T \gg \hbar \omega_0$). It makes use of the radiation pressure.

Two counter-propagating red detuned laser beams:

- **lab frame**: $+k, \omega_L v \quad -k, \omega_L$
- **c.o.m. frame**: $\omega_L - k.v \quad \omega_L + k.v$

The force is larger when the apparent frequency is closer to resonance

$\Rightarrow$ net force *against* the atomic velocity.
Laser cooling of the (classical) ion motion

(1) Doppler cooling

**Trap of frequency** $\omega_0$. Two successive steps.

(1) **Doppler cooling** on a broad transition $\Gamma \gg \omega_0$ ($\Gamma^{-1}$ lifetime of the electronic excited state) when the ion motion is still classical ($k_B T \gg \hbar \omega_0$). It makes use of the **radiation pressure**. Two counter-propagating **red detuned** laser beams:

- Lab frame: $+k, \omega_L$, $v$; $-k, \omega_L$
- Center of mass frame: $\omega_L - k.v$; $\omega_L + k.v$

The force is larger when the apparent frequency is closer to resonance

$\Rightarrow$ **net force against** the atomic velocity.

At low velocity we get a **friction force** $F = -\alpha v$ with a friction coefficient $\alpha > 0$ for $\omega_L < \omega_{eg}$. Final temperature: $k_B T \sim \hbar \Gamma \gg \hbar \omega_0$. $T \sim 100 \text{ mK}$
Laser cooling of the (quantum) ion motion

(2) Resolved sideband cooling

Recall: $H_I \simeq \hbar \frac{\Omega^2}{2} \left( e^{-i\omega t-i\varphi+ik\cdot \hat{r}\sigma_+} + e^{i\omega t+i\varphi-ik\cdot \hat{r}\sigma_-} \right)$

$N D \simeq \hbar \frac{\sigma}{\hbar} \left( a + a^+ \right) \sigma_+ + i\eta \left( a^+ \sigma_+ + i\sigma_+ a^+ \right) + i\eta \left( a \sigma_+ + \sigma_+ a \right)$

$H_c = \hbar \frac{\Omega^2}{2} (\sigma^+ + \sigma^-)$

$H_{BSB} = \eta \frac{\hbar \Omega^2}{2} (\sigma^+ \sigma^- + \sigma^- \sigma^+)$

$H_{RSB} = \eta \frac{\hbar \Omega^2}{2} (\sigma^+ \sigma^-)$
Laser cooling of the (quantum) ion motion

(2) Resolved sideband cooling

\[ H_c \]

\[ H_{\text{BSB}} : \text{blue sideband} \]

\[ H_{\text{RSB}} : \text{red sideband} \]

\[ \frac{i}{\hbar} \hbar \left( a^+ \sigma^+ + a \sigma^- \right) \]

\[ g_{n+1} \rightarrow e^+_{n+1} \]

\[ g_n \rightarrow e^-_{n+1} \]

\[ \frac{i}{\hbar} \hbar \left( a \sigma^+ + a^+ \sigma^- \right) \]
Laser cooling of the (quantum) ion motion

(2) Resolved sideband cooling

- To get to the ground state $n = 0$ of harmonic trap:
  use a narrow transition $\Gamma < \omega_0$.
- Transitions between vibrational states are now resolved.
  Coupling in the Lamb-Dicke regime $\eta = kx_0 \ll 1$:
  $$e^{ik\hat{x}}\sigma_+ = e^{i\eta(\hat{a} + \hat{a}^\dagger)}\sigma_+ \simeq [1 + \eta(\hat{a} + \hat{a}^\dagger)]\sigma_+$$
  carrier $(\sigma_+)$ at $\omega_{eg}$, red sideband $(\hat{a}\sigma_+)$ at $\omega_{eg} - \omega_0$,
  blue sideband $(\hat{a}^\dagger\sigma_+)$ at $\omega_{eg} + \omega_0$
- (1) Shine a laser on the $|g, n\rangle \rightarrow |e, n - 1\rangle$ transition.
- (2) In the Lamb-Dicke regime, spontaneous emission back to $|g\rangle$ preserves the vibrational state $|n\rangle_{ph}$:
  $|e, n - 1\rangle \rightarrow |g, n - 1\rangle$.
- Energy $\hbar\omega_0$ lost at each cycle.
- Repeat until $|g, 0\rangle$ (uncoupled!) is reached.
Resolved sideband cooling to the ground state

- spectrum after Doppler cooling: red sideband still visible
- spectrum after sideband cooling: only blue sideband remains, red sideband has disappeared
- In general, temperature can be estimated from the red/blue ratio \( P_{\text{red}} / P_{\text{blue}} = \langle n \rangle /(1 + \langle n \rangle) \)
  \[
  \langle n \rangle = \frac{1}{e^{\hbar \omega_0 / k_B T} - 1}
  \]
  \[
  \Rightarrow \frac{P_{\text{red}}}{P_{\text{blue}}} = e^{-\hbar \omega_0 / k_B T}
  \]

Fig. from Roos et al., PRL 83, 4713
The qubit: a two-level system

The strong Doppler cooling transition is not used for the qubit (decoherence due to spontaneous decay of state $|e\rangle$). Instead the two-level system used to define the qubit in ions can be of different nature:

- **Zeeman qubit**: between two Zeeman sublevels in the presence of a magnetic field ($\sim$ few MHz)
- **Hyperfine qubit**: between two hyperfine states ($\sim$ few GHz) driven by microwave or Raman transitions
- **Fine structure qubit**: between two states split by the fine structure ($\sim$ few THz)
- **Optical qubit**: between two stable electronic states, can be a ‘clock transition’ ($\sim$ few 100 MHz)

In this lecture we will mostly describe optical qubits.
Typical level structure

Different kinds of qubits:

(a) Basic Ion Structure (l=0, B=0)

(b) l=0 Ion Structure (Zeeman splitting)

(c) l≠0 Ion Structure (l=1/2, Hyperfine and Zeeman splitting)

Figure from Bruzewicz et al.
Typical level structure: Optical qubit

Ions used for quantum computation as well as clocks are typically alkaline earth atoms (Be\(^{+}\), Mg\(^{+}\), Ca\(^{+}\), Sr\(^{+}\)) or ytterbium Yb\(^{+}\).

**Optical qubit:** two **long-lived states** \(|0\rangle\) (ex: \(D_{5/2}\)) weekly coupled to the ground state \(|1\rangle\) (ex: \(S_{1/2}\))

Laser cooling and **detection:**

Short lived state \(|e\rangle\) (ex: \(P_{3/2}\)), strongly coupled to ground state \(|1\rangle\)

![Figure from Bruzewicz et al.](image-url)
Simplified level structure

The three operations performed on the ion state:

- **State preparation**
  - Figure from Bruzewicz et al.
  - For each long-lived electronic state $|0\rangle$ and $|1\rangle$, a ladder of phonon excitations $|n\rangle_{ph}$. Not relevant for $|e\rangle$ ($\Gamma \gg \omega_0$).

- **Qubit control**
  - $|0\rangle$ to $|1\rangle$ transition.

- **State detection**
  - $|1\rangle$ detection.
Qubit rotation (narrow line transition)

Carrier and sidebands

- Two states $|\uparrow\rangle \equiv |0\rangle$ and $|g\rangle \equiv |\downarrow\rangle \equiv |1\rangle$.
- Long lifetime $\Rightarrow$ no spontaneous decay, coherent (hamiltonian) evolution of the state.
- Coupling in the Lamb-Dicke regime $\eta = k x_0 \ll 1$:
  
  $$e^{i k \hat{x} \sigma_+} \simeq \left[1 + \eta (\hat{a} + \hat{a}^\dagger) \right] \sigma_+$$

- **carrier** ($\sigma_+$) at $\omega_{01}$: $|1, n\rangle \leftrightarrow |0, n\rangle$
- red sideband ($\hat{a}\sigma_+$) at $\omega_{01} - \omega_0$: $|1, n\rangle \leftrightarrow |0, n-1\rangle$
- blue sideband ($\hat{a}^\dagger\sigma_+$) at $\omega_{01} + \omega_0$:
  $|1, n\rangle \leftrightarrow |0, n+1\rangle$

Let’s describe in more detail the **carrier transition**, such that we can ignore the phonon state $|n\rangle_{ph}$. 
Qubit rotation: Narrow line transition at carrier frequency

Atomic system seen as a spin

Two states $|\uparrow\rangle \equiv |0\rangle$ (or $|0_z\rangle$) and $|g\rangle \equiv |\downarrow\rangle \equiv |1\rangle$ (or $|1_z\rangle$).

Operator basis set: Pauli operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$

and the other circular permutations

Spin lowering and raising operators

$$\sigma_+ = |\uparrow\rangle \langle \downarrow| = |0\rangle \langle 1| = \frac{\sigma_x + i\sigma_y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- = |\downarrow\rangle \langle \uparrow| = |1\rangle \langle 0| = \sigma_+^\dagger = \frac{\sigma_x - i\sigma_y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[\sigma_z, \sigma_{\pm}] = \pm 2\sigma_{\pm}$$
Laser manipulation of trapped ions

Qubit manipulation and detection

Arbitrary state of the two-level system

Most general observable $\sigma_u$ with $u = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$

$$
\sigma_u = \sigma \cdot u = \begin{pmatrix}
\cos \theta & \sin \theta e^{-i\phi} \\
\sin \theta e^{i\phi} & -\cos \theta 
\end{pmatrix}.
$$

Eigenvectors

$$
|\uparrow_u\rangle = |0_u\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle
$$

$$
|\downarrow_u\rangle = |1_u\rangle = -\sin \frac{\theta}{2} e^{-i\phi} |0\rangle + \cos \frac{\theta}{2} |1\rangle
$$

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Qubit manipulation and detection

Representation on the Bloch sphere

States $|0\rangle, |1\rangle, |0_x\rangle, |0_y\rangle, |0_u\rangle$:

State rotation $R_v(\alpha)$ by an angle $\alpha$ around the axis defined by $v$:

$$R_v(\alpha) = e^{-i(\alpha/2)\sigma_v} = \cos \frac{\alpha}{2} \mathbb{1} - i \sin \frac{\alpha}{2} \sigma_v$$

N.B. $R_v(2\pi) = -\mathbb{1}$

$|0_u\rangle, |1_u\rangle$ obtained by a rotation of $|0\rangle, |1\rangle$ of angle $\theta$ around the axis $v = (-\sin \phi, \cos \phi, 0)$. 
Rotation of states on the Bloch sphere

Around \(u_z\): \( R_z(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \)

Phase gate \(Z\): \( R_z(\pi) = -i \sigma_z = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

Around \(u_x\): \( R_x(\alpha) = \begin{pmatrix} \cos \frac{\alpha}{2} & -i \sin \frac{\alpha}{2} \\ -i \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \)

Spin flip \(X\): \( R_x(\pi) = -i \sigma_x = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

Hadamard gate \(H\):
\[
R_y\left(\frac{\pi}{2}\right) |0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = |0_x\rangle \\
R_y\left(\frac{\pi}{2}\right) |1\rangle = \frac{1}{\sqrt{2}} (-|0\rangle + |1\rangle) = |1_x\rangle
\]
Laser coupling at $\omega \approx \omega_{01}$

Hamiltonian ignoring external motion (carrier), detuning $\Delta = \omega - \omega_{01}$:

$$H = \frac{\hbar \omega_{01}}{2} \sigma_z + \frac{\hbar \Omega}{2} \left( e^{i \omega t + i \varphi} + e^{-i \omega t - \varphi} \right) (\sigma_+ + \sigma_-) = H'_0 - \frac{\hbar \Delta}{2} \sigma_z + H_I$$

Interaction representation with respect to $H'_0 = \hbar \omega \sigma_z / 2$ inducing a spin precession at the field frequency $\omega$.

Hamiltonian for $|\tilde{\psi}\rangle = U'_0^\dagger |\psi\rangle$ (hence $|\psi\rangle = U'_0 |\tilde{\psi}\rangle$) with

$U'_0^\dagger = \exp(iH'_0 t / \hbar)$ i.e. $U'_0 = \exp(-iH'_0 t / \hbar) = e^{-i \frac{\omega t}{2} \sigma_z} = R_z(\omega t)$:

$$H' = \hbar \omega \sigma_z$$

$U'_0^\dagger = \psi'_0$ $H'_0$

$U'_0 = \tilde{\psi}$ where $H = U'_0^\dagger (-\frac{\hbar \Delta}{2} \sigma_z + \hbar \omega \sigma_z) U'_0$

$\sigma'_\pm = U'_0^\dagger \sigma_\pm U'_0$

$\tilde{\sigma}'_\pm |\psi\rangle = e^{\frac{i \omega t}{2} \sigma_z} e^{-\frac{i \omega t}{2} \sigma_\pm} |\psi\rangle = e^{\frac{i \omega t}{2} \sigma_z} e^{-\frac{i \omega t}{2} \sigma_\pm} |\psi\rangle$
Laser coupling at $\omega \simeq \omega_{01}$

Hamiltonian ignoring external motion (carrier), detuning $\Delta = \omega - \omega_{01}$:

$$H = \frac{\hbar \omega_{01}}{2} \sigma_z + \frac{\hbar \Omega}{2} \left( e^{i\omega t} + e^{-i\omega t} \right) \left( \sigma_+ + \sigma_- \right) = H'_0 - \frac{\hbar \Delta}{2} \sigma_z + H_I$$

Interaction representation with respect to $H'_0 = \hbar \omega \sigma_z / 2$ inducing a spin precession at the field frequency $\omega$.

Hamiltonian for $|\tilde{\psi}\rangle = U_0^{\dagger} |\psi\rangle$ with $U'_0 = e^{-iH'_0 t / \hbar} = e^{-i \frac{\omega t}{2} \sigma_z} = R_z(\omega t)$:

$$\tilde{H} = U'_0^{\dagger} \left( -\frac{\hbar \Delta}{2} \sigma_z + H_I \right) U'_0$$

$\sigma_z$ part of $H_0$ unchanged (commutes with the evolution operator) but

$$\tilde{\sigma}_\pm = U'_0^{\dagger} \sigma_\pm U'_0 = e^{i \frac{\omega t}{2} \sigma_z} \sigma_\pm e^{-i \frac{\omega t}{2} \sigma_z} = e^{\pm i\omega t} \sigma_\pm.$$
Spin rotation at $\omega \simeq \omega_{01}$

Rabi precession

$$\tilde{H} = -\frac{\hbar}{2} \sigma_z + \frac{\hbar}{2} \left( e^{-i(\omega t + \varphi)} + e^{i(\omega t + \varphi)} \right) \left( e^{i\omega t} \sigma_+ + e^{-i\omega t} \sigma_- \right)$$

Two rapidly oscillating terms, and two constant ones.

Rotating wave approximation (RWA): neglect terms oscillating rapidly in $\tilde{H}$

$$\tilde{H} = -\frac{\hbar}{2} \sigma_z + \frac{\hbar}{2} \left( \sigma_+ e^{-i\varphi} + \sigma_- e^{i\varphi} \right)$$

$$= -\frac{\hbar}{2} \sigma_z + \frac{\hbar}{2} \left( \sigma_x \cos \varphi + \sigma_y \sin \varphi \right) = \frac{\hbar}{2} \sigma_n$$

with $\mathbf{n} = \frac{-\Delta \mathbf{u}_z + \Omega \cos \varphi \mathbf{u}_x + \Omega \sin \varphi \mathbf{u}_y}{\Omega'}$ and $\Omega' = \sqrt{\Omega^2 + \Delta^2}$.

Hence, evolution operator $U(t) = e^{-i \frac{\Omega' t}{2} \sigma_n} = R_n(\Omega' t)$. Spin rotation!

To flip the spin completely, $\mathbf{n}$ should be in the horizontal plane $\Rightarrow \Delta = 0$. 
Spin rotation at $\omega \simeq \omega_{01}$

Rabi precession

$$\tilde{H} = \frac{\hbar \Omega'}{2} \sigma_n$$

where

$$\Omega' = \sqrt{\Omega^2 + \Delta^2}$$

$$n = \cos \theta u_z + \sin \theta (\cos \varphi u_x + \sin \varphi u_y)$$

with $\cos \theta = -\Delta/\Omega'$ and $\sin \theta = \Omega/\Omega'$. 
Resonant case $\Delta = 0$: Spin rotation at $\omega = \omega_{01}$

Rabi precession

Resonant case $\Delta = 0$: rotation around an axis in the equatorial plane $\mathbf{n} = \cos \varphi \mathbf{u}_x + \sin \varphi \mathbf{u}_y$. Choosing $|g\rangle = |1\rangle$ as the initial state

$$|\tilde{\psi}(t)\rangle = -i e^{-i\varphi} \sin \frac{\Omega t}{2} |0\rangle + \cos \frac{\Omega t}{2} |1\rangle$$

$$p_0(t) = \frac{1 - \cos(\Omega t)}{2}$$  

Rabi oscillation at frequency $\Omega$.

More generally, evolution operator for $|\tilde{\psi}\rangle$:

$$U(t) = \begin{pmatrix} \cos \frac{\Omega t}{2} & -i e^{-i\varphi} \sin \frac{\Omega t}{2} \\ -i e^{i\varphi} \sin \frac{\Omega t}{2} & \cos \frac{\Omega t}{2} \end{pmatrix}$$
Laser manipulation of trapped ions Qubit manipulation and detection

Resonant case $\Delta = 0$: Spin rotation at $\omega = \omega_{01}$

Rabi precession

Some particular choices of pulse duration:

- ‘$\pi/2$ pulse’, i.e. $t = \pi/2\Omega$. Evolution operator

$$R_n(\pi/2) = \frac{1}{\sqrt{2}} (1 - i\sigma_n) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -ie^{-i\varphi} \\ -ie^{i\varphi} & 1 \end{pmatrix}$$

Recover Hadamard gate for $\varphi = \pi/2$: 

$$\left\{ \begin{array}{c} |0\rangle \longrightarrow \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\ |1\rangle \longrightarrow \frac{1}{\sqrt{2}} (-|0\rangle + |1\rangle) \end{array} \right.$$  

- ‘$\pi$-pulse’: $\Omega t = \pi$ with $\varphi = 0$. $R_x(\pi) = -i\sigma_x \Rightarrow$ Spin flip X-gate

- ‘$2\pi$ pulse’: $\Omega t = 2\pi$ global $-$ sign associated to a $2\pi$ rotation of a spin-1/2.

N.B. The Z-gate ($\pi$-rotation around Z, corresponding to $|\Delta| \gg \Omega$) is rather obtained by shifting the laser phase $\varphi$ by $\pi$ before next pulse.
Qubit rotation: Narrow line transition at sideband frequencies

Near a sideband transition, the two coupled states have different phonon excitations.

- **Blue sideband**: in the interaction picture and within RWA, coupling \( \eta \Omega \sqrt{n + 1} \)

\[
H_I = \frac{\hbar \eta \Omega}{2} \left( \hat{a}^\dagger \sigma_+ e^{-i\varphi} + \hat{a} \sigma_- e^{i\varphi} \right)
\]

- Couples state \(|1\rangle|0\rangle_{ph}\) with \(|0\rangle|1\rangle_{ph}\). X-gate, Z-gate, Hadamard \(H\)-gate possible between these states.

Example of \(\pi/2\) pulse:

\[
|0\rangle|1\rangle_{ph} \quad \rightarrow \quad \frac{1}{\sqrt{2}} \left( |0\rangle|1\rangle_{ph} + |1\rangle|0\rangle_{ph} \right)
\]

\[
|1\rangle|0\rangle_{ph} \quad \rightarrow \quad \frac{1}{\sqrt{2}} \left( -|0\rangle|1\rangle_{ph} + |1\rangle|0\rangle_{ph} \right)
\]

- BSB does **nothing** on \(|0\rangle|0\rangle_{ph}\) state!
Qubit rotation: Narrow line transition at sideband frequencies

Near a sideband transition, the two coupled states have different phonon excitations.

- **Red sideband**: in the interaction picture and within RWA, coupling $\eta \Omega \sqrt{n + 1}$

\[
H_I = \frac{\hbar \eta \Omega}{2} \left( \hat{a} \sigma_+ e^{-i\varphi} + \hat{a}^\dagger \sigma_- e^{i\varphi} \right)
\]

- Couples state $|1\rangle |1\rangle_{ph}$ with $|0\rangle |0\rangle_{ph}$. $X$-gate, $Z$-gate, Hadamard $H$-gate possible between these states.

Example of $\pi/2$ pulse:

\[
|0\rangle |0\rangle_{ph} \quad \rightarrow \quad \frac{1}{\sqrt{2}} \left( |0\rangle |0\rangle_{ph} + |1\rangle |1\rangle_{ph} \right)
\]

\[
|1\rangle |1\rangle_{ph} \quad \rightarrow \quad \frac{1}{\sqrt{2}} \left( -|0\rangle |0\rangle_{ph} + |1\rangle |1\rangle_{ph} \right)
\]

- RSB does nothing on $|1\rangle |0\rangle_{ph}$ state!
Qubit rotation: Experimental results

Carrier Rabi oscillations and blue sideband Rabi oscillations for a Ca\(^{+}\) ion: different timescales. \(S_{1/2} \rightarrow D_{5/2}\) line at 729 nm. Innsbruck group.

carrier
\[ 2\pi/\Omega = 11 \ \mu s \]

BSB
\[ 2\pi/(\eta \Omega \sqrt{n + 1}) = 270 \ \mu s \]
Detection of the qubit state (strong transition)

Electron shelving method

$|1\rangle$ strongly coupled to the excited state $|e\rangle$ (e.g. Doppler cooling transition). Photons scattering when driving the $|1\rangle \rightarrow |e\rangle$ transition. $|1\rangle$ is the **bright state** while the metastable $|0\rangle$ state remains **dark**.

State detection

Fluorescence signal

Observation of quantum jumps
Joint detection of the qubit and phonon state

Method valid for $n = 0, 1$

How to detect the state $|x\rangle|n_{\text{ph}}\rangle$? Repeat two kinds of measurements:

1. Search for $|0\rangle|n_{\text{ph}}\rangle$:

<table>
<thead>
<tr>
<th>state</th>
<th>fluo</th>
<th>after BSB-$\pi$</th>
<th>fluo</th>
<th>signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>0\rangle</td>
<td>0\rangle_{\text{ph}}$</td>
<td>dark</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>0\rangle</td>
<td>1\rangle_{\text{ph}}$</td>
<td>dark</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>1\rangle</td>
<td>n\rangle_{\text{ph}}$</td>
<td>bright</td>
<td>-</td>
</tr>
</tbody>
</table>

2. Search for $|1\rangle|n_{\text{ph}}\rangle$:

<table>
<thead>
<tr>
<th>state</th>
<th>after carrier-$\pi$</th>
<th>fluo</th>
<th>after BSB-$\pi$</th>
<th>fluo</th>
<th>signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>0\rangle</td>
<td>0\rangle_{\text{ph}}$</td>
<td>$</td>
<td>1\rangle</td>
<td>0\rangle_{\text{ph}}$</td>
</tr>
<tr>
<td>$</td>
<td>0\rangle</td>
<td>1\rangle_{\text{ph}}$</td>
<td>$</td>
<td>1\rangle</td>
<td>1\rangle_{\text{ph}}$</td>
</tr>
<tr>
<td>$</td>
<td>1\rangle</td>
<td>0\rangle_{\text{ph}}$</td>
<td>$</td>
<td>0\rangle</td>
<td>0\rangle_{\text{ph}}$</td>
</tr>
<tr>
<td>$</td>
<td>1\rangle</td>
<td>1\rangle_{\text{ph}}$</td>
<td>$</td>
<td>0\rangle</td>
<td>1\rangle_{\text{ph}}$</td>
</tr>
</tbody>
</table>
Joint detection of the qubit and phonon state

Experimental results: Truth table

with trapped, $^{88}\text{Sr}^+$, from J. Łabaziewicz PhD thesis (MIT, 2008)
Effect of a carrier-$\pi$ pulse

Coupling: $\Omega(\sigma_+ + \sigma_-) \Rightarrow \pi$-pulse for $\Omega t = \pi$

with trapped, $^{88}$Sr$^+$, from J. Łabaziewicz PhD thesis (MIT, 2008)
Effect of a BSB-$\pi$ pulse

Coupling: $\Omega(\hat{a}^\dagger \sigma_+ + \hat{a} \sigma_-)$ does nothing on $|0\rangle|0\rangle_{ph}$ state

$\Omega a^\dagger |0\rangle_{ph} = \Omega |1\rangle_{ph} \Rightarrow \pi$-pulse on $|1\rangle|0\rangle_{ph} \leftrightarrow |0\rangle|1\rangle_{ph}$ for $\Omega t = \pi$

And for $|1\rangle|1\rangle_{ph} \rightarrow |0\rangle|2\rangle_{ph}$?
Effect of a BSB-\(\pi\) pulse

Coupling: \(\Omega(\hat{a}^\dagger \sigma_+ + \hat{a} \sigma_-)\) does nothing on \(|0\rangle|0\rangle_{ph}\) state
\[\Omega \hat{a}^\dagger |0\rangle_{ph} = \Omega |1\rangle_{ph} \Rightarrow \pi\)-pulse on \(|1\rangle |0\rangle_{ph} \leftrightarrow |0\rangle |1\rangle_{ph}\) for \(\Omega t = \pi\)

And for \(|1\rangle |1\rangle_{ph} \rightarrow |0\rangle |2\rangle_{ph}\)?
\[\Omega \hat{a}^\dagger |1\rangle_{ph} = \Omega \sqrt{2} |2\rangle_{ph}\]
\(\Rightarrow\) not a \(\pi\)-pulse

\[
\begin{pmatrix}
1.6 & 2.0 & 3.6 & 93.2 \\
5.9 & 95.1 & 23.8 & 1.6 \\
1.8 & 0.6 & 37.2 & 3.7 \\
91.7 & 3.9 & 35.4 & 2.7
\end{pmatrix}
\]

\(|1\rangle |1\rangle_{ph} \rightarrow \alpha |1\rangle |1\rangle_{ph} + \beta |0\rangle |2\rangle_{ph}\)
not properly measured

with trapped, \(^{88}\text{Sr}^+\), from J. Łabaziewicz PhD thesis (MIT, 2008)
Coupling ions via phonons
Using phonons as control qubit in a two-qubit gate

- Some transitions (BSB, RSB) are not allowed depending on the phonon excitation ⇒ use this for 2-qubit gates with a control qubit.
- Possible with 1 ion (control = phonon, target = internal state) using an auxiliary internal state $|g'\rangle$ (or a subtle pulse sequence):

\[
\begin{array}{c}
\circ \\

\circ |g'\rangle
\end{array}
\]

initial state
Using phonons as control qubit in a two-qubit gate

- Some transitions (BSB, RSB) are not allowed depending on the phonon excitation ⇒ use this for 2-qubit gates with a control qubit.
- Possible with 1 ion (control = phonon, target = internal state) using an auxiliary internal state $|g'\rangle$ (or a subtle pulse sequence):

\[
\begin{align*}
|g\rangle & \quad \text{(initial state)} \\
|g\rangle & +\pi/2 - \text{carrier}
\end{align*}
\]
Using phonons as control qubit in a two-qubit gate

- Some transitions (BSB, RSB) are not allowed depending on the phonon excitation ⇒ use this for 2-qubit gates with a control qubit.
- Possible with 1 ion (control = phonon, target = internal state) using an auxiliary internal state $|g'\rangle$ (or a subtle pulse sequence):
Using phonons as control qubit in a two-qubit gate

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- Possible with 1 ion (control = phonon, target = internal state) using an auxiliary internal state $|g'\rangle$ (or a subtle pulse sequence):

\[
\begin{align*}
&\text{initial state} \quad |g'\rangle \\
&+\pi/2\text{-carrier} \\
&2\pi \text{ aux. BSB} \\
&-\pi/2\text{-carrier}
\end{align*}
\]

CNOT gate: second $-\pi/2$-pulse on carrier erases the effect of the first, unless a phase change has occurred with the BSB pulse to $|g'\rangle$.

\[
\begin{align*}
|x\rangle |0\rangle_{ph} &\rightarrow |x\rangle |0\rangle_{ph} \\
|x\rangle |1\rangle_{ph} &\rightarrow |\text{NOT}(x)\rangle |1\rangle_{ph}
\end{align*}
\]
Coupling ions via phonons

- Some transitions (BSB, RSB) are not allowed depending on the phonon excitation ⇒ use this for 2-qubit gates with a control qubit.
- Possible with 1 ion (control = phonon, target = internal state)
- Even better with several ions, to implement algorithms with many qubits
- Ion chain in a linear Paul trap: collective motional modes
- The phonon excitation is shared by all ions

Ex with two ions in harmonic trap + Coulomb repulsion:

center of mass mode at $\omega_0$

breathing mode at $\sqrt{3}\omega_0$
Normal modes of an ion chain

N ions, \( x_i \), \( x_1 < x_2 < \cdots < x_N \)

Two forces: 
- \(-m\omega_0^2 x_i\) - Coulomb force: \( + \frac{q^2}{4\pi\varepsilon_0 (x_i - x_j)^2} \)

Coupled equations:
\[
\begin{align*}
m\ddot{x}_i &= -m\omega_0^2 x_i + \sum_{j<i} \frac{q^2}{4\pi\varepsilon_0 (x_i - x_j)^2} - \sum_{j>i} \frac{q^2}{4\pi\varepsilon_0 (x_i - x_j)^2}
\end{align*}
\]

\( \rightarrow \) collective mode

Center of mass mode: \( X = \frac{1}{N} \sum_{i=1}^{N} x_i \)
\[
mN \ddot{X} = -mN\omega_0^2 X + \frac{q^2}{4\pi\varepsilon_0} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{1}{(x_i - x_j)^2}
\]

\( \ddot{X} = -\omega_0^2 X \)

Expand coupled eq. close to equilibrium, \( \rightarrow \) normal modes. Ex: \( x_i - x_i^{eq} \), \( \omega_0^2 = 3\omega_0^2 \)
Normal modes of an ion chain

Ex with 7 ions (H-C Nägerl thesis 1998):

- Center of mass mode at $\omega_0$
- Breathing mode at $\sqrt{3}\omega_0$
- etc.
CNOT gate with two ions coupled by phonons

With laser-addressable ions

- The two ions share the same c.o.m. mode.
- Initialization in the common ground state of motion $|0\rangle_{\text{ph}}$.

\[
\begin{align*}
|0\rangle|0\rangle|1\rangle_{\text{ph}} & \quad \longrightarrow \quad |0\rangle|0\rangle|0\rangle_{\text{ph}} \\
|0\rangle|1\rangle|1\rangle_{\text{ph}} & \quad \longrightarrow \quad |1\rangle|0\rangle|1\rangle_{\text{ph}} \\
|0\rangle|1\rangle|0\rangle_{\text{ph}} & \quad \longrightarrow \quad |1\rangle|0\rangle|0\rangle_{\text{ph}} \\
|1\rangle|1\rangle|0\rangle_{\text{ph}} & \quad \longrightarrow \quad |1\rangle|1\rangle|1\rangle_{\text{ph}}
\end{align*}
\]
CNOT gate with two ions coupled by phonons

With laser-addressable ions

- The two ions share the same c.o.m. mode.
- Initialization in the common ground state of motion \(|0\rangle_{\text{ph}}\).
- \(\pi\)-BSB on ion 1 (control), only if in \(|1\rangle\): \(|1\rangle |x\rangle |0\rangle_{\text{ph}} \leftrightarrow |0\rangle |x\rangle |1\rangle_{\text{ph}}

\[
\begin{align*}
|0\rangle |0\rangle |1\rangle_{\text{ph}} & \quad |0\rangle |0\rangle |0\rangle_{\text{ph}} \\
|0\rangle |1\rangle |1\rangle_{\text{ph}} & \quad |1\rangle |0\rangle |1\rangle_{\text{ph}} \\
|0\rangle |1\rangle |0\rangle_{\text{ph}} & \quad |1\rangle |0\rangle |0\rangle_{\text{ph}} \\
|1\rangle |1\rangle |0\rangle_{\text{ph}} & \quad |1\rangle |1\rangle |1\rangle_{\text{ph}}
\end{align*}
\]
CNOT gate with two ions coupled by phonons

With laser-addressable ions

- The two ions share the same c.o.m. mode.
- Initialization in the common ground state of motion $|0\rangle_{ph}$.
- $\pi$-BSB on ion 1 (control), only if in $|1\rangle$: $|1\rangle|x\rangle|0\rangle_{ph} \leftrightarrow |0\rangle|x\rangle|1\rangle_{ph}$
- CNOT on ion 2 controlled by motion (see above).
CNOT gate with two ions coupled by phonons

With laser-addressable ions

- The two ions share the same c.o.m. mode.
- Initialization in the common ground state of motion $|0\rangle_{ph}$.
- $\pi$-BSB on ion 1 (control), only if in $|1\rangle$: $|1\rangle |x\rangle |0\rangle_{ph} \leftrightarrow |0\rangle |x\rangle |1\rangle_{ph}$
- CNOT on ion 2 controlled by motion (see above).
- $\pi$-BSB on ion 1: back to initial state for ion 1.

The scheme also works with $N > 2$ addressable ions.
Mølmer-Sørensen gate

Preparation of a Bell state

- Proposed by K. Mølmer and A. Sørensen, PRL 82, 1971 (1999)
- Allows to prepare directly a Bell state \(|00\rangle - |11\rangle)/\sqrt{2}
- Two-photon coupling from \(|00\rangle\) to \(|11\rangle\)
- \(\omega_r = \omega_{01} - \omega_0 - \Delta, \omega_b = \omega_{01} + \omega_0 + \Delta\)
- Interference between paths \(\Rightarrow\) coupling independent of \(n\):
  \[\Omega_{MS} = 2 \times \left[ (n + 1) \frac{\Omega_b \Omega_r}{2\Delta} + n \frac{\Omega_r \Omega_b}{-2\Delta} \right] = \frac{\Omega_r \Omega_b}{\Delta}\]

\[
\begin{align*}
|00\rangle|n\rangle_{ph} & \quad \sqrt{n+1}\Omega_r \quad \sqrt{n}\Omega_b \quad \sqrt{n}\Omega_r \quad \sqrt{n+1}\Omega_b \\
|01\rangle|n+1\rangle_{ph} & \quad |01\rangle|n\rangle_{ph} \quad |01\rangle|n-1\rangle_{ph} \\
|10\rangle|n+1\rangle_{ph} & \quad |10\rangle|n\rangle_{ph} \quad |10\rangle|n-1\rangle_{ph}
\end{align*}
\]

NB: \(|01\rangle\) and \(|10\rangle\) are coupled as well.
### State-of-the-art single-qubit gate demonstrations

<table>
<thead>
<tr>
<th>Type</th>
<th>Method</th>
<th>Fidelity</th>
<th>Time (µs)</th>
<th>Species</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
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<td>5</td>
<td>$^{40}\text{Ca}^+$</td>
<td>Bermudez 2017</td>
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<tr>
<td></td>
<td>Raman</td>
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<td>7.5</td>
<td>$^{43}\text{Ca}^+$</td>
<td>Ballance 2016</td>
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<td>$^{9}\text{Be}^+$</td>
<td>NIST 2016</td>
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<tr>
<td></td>
<td>Raman</td>
<td>0.99</td>
<td>0.00005</td>
<td>$^{171}\text{Yb}^+$</td>
<td>Campbell 2010</td>
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<tr>
<td></td>
<td>Raman</td>
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<td>$^{88}\text{Sr}^+$</td>
<td>Keselman 2011</td>
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<tr>
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<td>µwave</td>
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<td>$^{43}\text{Ca}^+$</td>
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<td></td>
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<td>0.0186</td>
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<td>$^{25}\text{Mg}^+$</td>
<td>Ospelkaus 2011</td>
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</tbody>
</table>

Adapted from Bruzewicz et al.
State-of-the-art two-qubit gate demonstrations

<table>
<thead>
<tr>
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<th>Method</th>
<th>Fidelity</th>
<th>Time (µs)</th>
<th>Species</th>
<th>Ref.</th>
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<tbody>
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<td>(1 sp.)</td>
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<td>0.993</td>
<td>50</td>
<td>( ^{40}\text{Ca}^+ )</td>
<td>Benhelm 2008</td>
</tr>
<tr>
<td></td>
<td>Raman</td>
<td>0.9991(6)</td>
<td>30</td>
<td>( ^{9}\text{Be}^+ )</td>
<td>NIST 2016</td>
</tr>
<tr>
<td></td>
<td>Raman</td>
<td>0.999</td>
<td>100</td>
<td>( ^{43}\text{Ca}^+ )</td>
<td>Ballance 2016</td>
</tr>
<tr>
<td></td>
<td>Raman</td>
<td>0.998</td>
<td>1.6</td>
<td>( ^{43}\text{Ca}^+ )</td>
<td>Schafer 2018</td>
</tr>
<tr>
<td></td>
<td>( \mu \text{wave} )</td>
<td>0.60</td>
<td>0.5</td>
<td>( ^{43}\text{Ca}^+ )</td>
<td>Schafer 2018</td>
</tr>
<tr>
<td></td>
<td>( \mu \text{wave} )</td>
<td>0.997</td>
<td>3250</td>
<td>( ^{43}\text{Ca}^+ )</td>
<td>Harty 2016</td>
</tr>
<tr>
<td></td>
<td>( \mu \text{wave} )</td>
<td>0.985</td>
<td>2700</td>
<td>( ^{171}\text{Yb}^+ )</td>
<td>Weidt 2017</td>
</tr>
<tr>
<td>2-qubit</td>
<td>Ram./Ram.</td>
<td>0.998(6)</td>
<td>27.4</td>
<td>( ^{40}\text{Ca}^+/^{43}\text{Ca}^+ )</td>
<td>Ballance 2015</td>
</tr>
<tr>
<td>(2 sp.)</td>
<td>Ram./Ram.</td>
<td>0.979(1)</td>
<td>35</td>
<td>( ^{9}\text{Be}^+/^{25}\text{Mg}^+ )</td>
<td>Tan 2015</td>
</tr>
</tbody>
</table>

Adapted from Bruzewicz et al.
Algorithms performed with trapped ions

Results of Grover algorithm, Figatt et al., Nature Comm. 8, 1918 (2017)
Algorithms implemented with trapped ions

The recent implementations were done by Monroe’s group (Maryland) or Blatt’s group (Innsbruck).

- **Spin and phase flip error detection** up to 7 ions [Nigg et al. Science 345, 302 (2014)]

- **Deutsch-Jozsa / Bernstein-Vazirani** algorithm: determines if a $N \rightarrow 1$ operation is either constant or balanced ($3 + 1$ ions + ancilla) [Debnath et al. Nature 536, 63 (2016)]

- **Quantum Fourier transform (QFT)** (with 5 ions) [Debnath et al. ibid.]

- **Grover’s search** algorithm (with 2 and 3 ions) [C. Figgatt et al., Nature Comm. 8, 1918 (2017)]

- **Shor** algorithm (with 5 ions): $15 = 3 \times 5$ [Monz et al. Science 351, 1068 (2016)]

- **Variational quantum energy solver (VQE)** applied to the ground state of H$_2$, LiH and H$_2$O (up to 11 ions) [C. Hempel et al., Phys. Rev. X 8, 031022 (2018); Y. Nam et al., arXiv:1902.10171 (2019)]
Example: Deutsch-Jozsa algorithm
Implementation with 95% success rate on a 5-qubit computer

Experimental setup [Debnath et al. Nature 536, 63 (2016)].

- 5 $^{171}\text{Yb}^+$ ions
- Paul trap $3 \times 0.27$ MHz
- spacing 5 $\mu$m
- HF states $|F = 0, m_F = 0\rangle$ and $|F = 1, m_F = 0\rangle$, 12.6 GHz
- gates: Raman transitions
- $\tau_{coh} > 0.5$ s
- single qubit detection $> 99%$
- 5 qubits: 95.3% on average
- Deutsch-Jozsa: $|x\rangle$ with 3 qubits, 1 qbit for register, 1 ancilla
Example: Deutsch-Jozsa algorithm

Implementation with 95% success rate on a 5-qubit computer

Algorithm:

1. Prepare equal superposition of all $|x\rangle$ and ancilla in
   $|a\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$
2. Use $C_i\text{NOT}(4)$ to apply $f(x)$:
   $\sum_x |x\rangle |f(x)\rangle |a\rangle$
3. $C_4\text{NOT}(5)$
   $\sum_x (-1)^{f(x)} |x\rangle |f(x)\rangle |a\rangle$
4. Last $R_y$ brings the state in
   $(|y\rangle |0\rangle |0\rangle + |y'\rangle |1\rangle |0\rangle)/\sqrt{2}$
   such that $f(x) = x \cdot \bar{y}$
5. Conditional meas. $X_4 = 0$: get $|y\rangle$ or $|111\rangle$
DiVincenzo criteria with a trapped-ion quantum computer

- A scalable physical system with well characterised qubits. **Scalability?**
- The ability to initialize the state of the qubits to a simple fiducial state. **OK with laser cooling and optical pumping, < 1 ms**
- Long relevant decoherence times. **OK with microwave transitions or narrow-line optical transitions, > 100 ms**
- A universal set of quantum gates. **OK: single-qubit rotations with fidelity up to 99.9999% (microwave), CNOT up to 99.9%**
- A qubit-specific measurement capability. **OK with the shelving method, < 200 µs with fidelity 99.99%**

N.B. State preparation + readout with a fidelity 99.93% demonstrated, i.e. state preparation error $\sim 2 \times 10^{-4}$.

Source: Bruzewicz et al.
Efforts towards scalability

- Development of surface ion traps
- Having hundreds of addressable ions in a trap is ‘easy’
- Difficult point: controlling their interaction (with phonons)
- Possible solution: displace ions in a specific interaction zone
- Explored by Dave Wineland (NIST Boulder), Chris Monroe (Univ. Maryland and IonQ company), . . .

Fig. from Bruzewicz et al.

Fig. from NIST
**Progress in optics integration**

**Fig. from Niffenegger et al., Nature 586, 538 (2020) / arXiv:2001:05052**
Bibliography for Lecture 1


- C. Champenois, *Trapping and cooling of ions*, Les Houches lecture notes https://cel.archives-ouvertes.fr/cel-00334152


- S. Debnath et al., *Demonstration of a small programmable quantum computer with atomic qubits*, Nature 536, 63 (2016)