Low-dimensional Bose gases
Part 2: phase diagrams of low D systems

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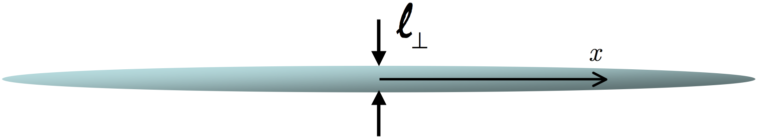
Outline

Outline of the lecture

1. 1D Bose gas
   - Regimes of the 1D Bose gas at $T = 0$
   - Thermal phase fluctuations
   - Tonks-Girardeau gas
2. 2D Bose gas
   - Uniform interacting 2D gas
   - Berezinskii-Kosterlitz-Thouless transition
   - Trapped interacting 2D gas

1D Bose gas

1D interacting Bose gas



Phases of the 1D Bose gas

N.B. stay in the regular case $a \ll \ell_\perp$, $g_1 = 2\hbar\omega_\perp a$

- interaction parameter $\gamma = \dfrac{Mg_1}{\hbar^2 n} = \dfrac{2a}{n\ell_\perp^2}$
- trapped gas

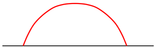
  Longitudinal trapping along x with frequency ω_x, $\ell_x = \sqrt{\frac{\hbar}{M\omega_x}}$

  $$\text{new parameter} \quad \alpha = \frac{g_1 \ell_x^{-1}}{\hbar\omega_x} = \frac{2a\ell_x}{\ell_\perp^2} = \gamma n \ell_x > \gamma$$
- identify 3 regimes: Gaussian BEC, Thomas-Fermi BEC ($\mu \gg \hbar\omega_x$), Tonks gas ($\gamma \gg 1$)

Weak interactions

Weak interactions $\gamma \ll 1$: true BEC at $T = 0$ thanks to the $\hbar\omega_x$ low energy cut-off

- Thomas-Fermi regime $\mu \gg \hbar\omega_x$ and $\gamma < 1$:



$$\mu = \frac{\hbar\omega_x}{2}\left(\frac{3N\alpha}{2}\right)^{2/3} \qquad R_{\mathrm{TF}} = \ell_x\left(\frac{3N\alpha}{2}\right)^{1/3}$$

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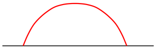

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$$n \sim \frac{N}{R_{\mathrm{TF}}} \Longrightarrow \gamma \sim \left(\frac{\alpha^2}{N}\right)^{2/3}$$

weak interactions $\gamma < 1 \Longrightarrow N > N^*$, with $N^* = \alpha^2$

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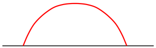

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- $N < \alpha^{-1}$, $\gamma < \alpha \ll 1$: Gaussian BEC (strong trap, few atoms)

Strong interactions

Strong interactions $\gamma > 1$: implies $\alpha \gg 1$ and $N < N^*$ (weak trap, few atoms) $\Longrightarrow$ Tonks gas

- uniform 1D gas: Lieb-Liniger model

$$H = \sum_{i=1}^{N} -\frac{\hbar^2}{2M}\partial_{x_i}^2 + g_1 \sum_{i<j} \delta(x_i - x_j)$$

exactly solvable: $\mu = \dfrac{\pi^2\hbar^2 n^2}{2M}\left(1 - \dfrac{8}{3\gamma}\right)$ for $\gamma \gg 1$

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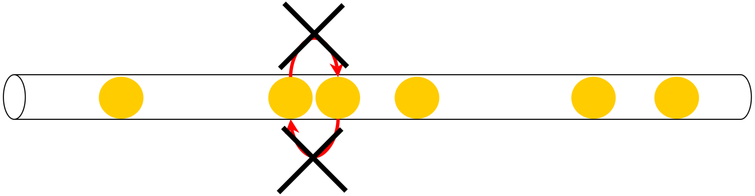
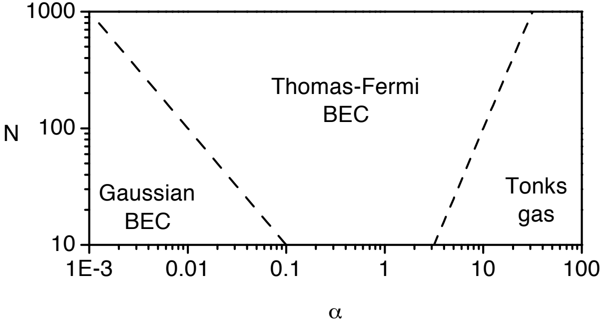
- $\gamma \longrightarrow \infty$ limit: Tonks-Girardeau gas of impenetrable bosons



total reflection at each collision (scattering amplitude $f = -1$)

3 regimes of interacting 1D Bose gas at $T = 0$



Limits: $N = \alpha^{-1}$ and $N = \alpha^2$

From Petrov, Gangardt, Shlyapnikov, QGLD 2003 Proceedings

Thermal phase fluctuations in the 1D Bose gas

Now $T > 0$, weak interacting regime $\gamma \ll 1$

- thermal density fluctuations are small; TF profile for $T < T_d = N\hbar\omega_x$, with $R_{\mathrm{TF}} = \ell_x \left(\frac{3N\alpha}{2}\right)^{1/3}$

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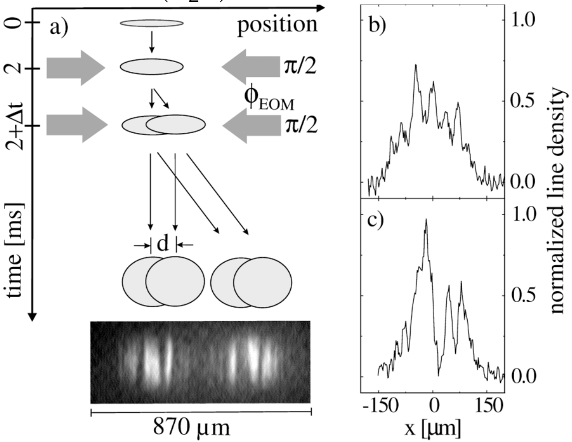
$\langle \delta\phi^2(x) \rangle \propto \frac{|x|}{\ell_\phi}$

Hellweg et al., 2003



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$$T_\phi = T_d \frac{\hbar\omega_x}{\mu} = \frac{T_d}{(N\alpha)^{2/3}} \ll T_d$$

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$$T_\phi = T_d \frac{\hbar\omega_x}{\mu} = \frac{T_d}{(N\alpha)^{2/3}} \ll T_d$$

- exponential decay of $g^{(1)}(x) = \exp(-\frac{|x|}{\ell_\phi})$
  $\Longrightarrow$ Lorentzian momentum distribution

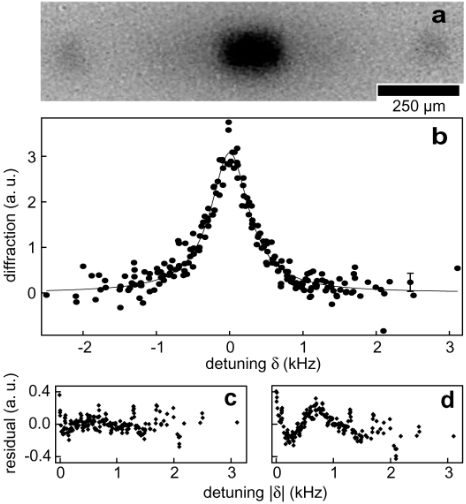
Thermal phase fluctuations in the 1D Bose gas

Momentum distribution in a 1D quasi-condensate
Richard et al., 2003

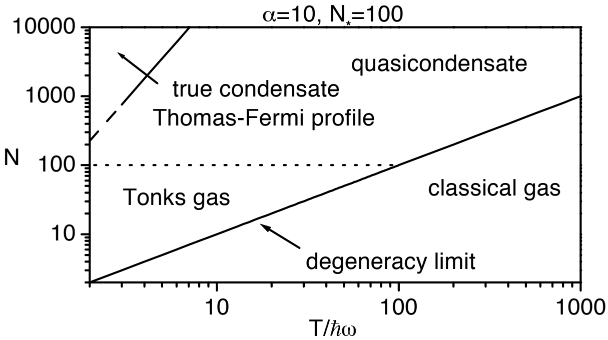
a: Bragg scattering technique

b: Lorentzian distribution for p_x

c/**d**: comparison Lorentzian/Gaussian fit



Summary: diagram of states at finite T



Limits: $T = T_d$, $T = T_\phi$ and $N = N^*$

From Petrov, Gangardt, Shlyapnikov, QGLD 2003 Proceedings

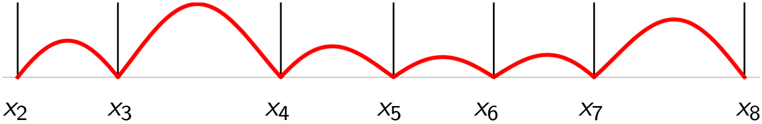
Tonks-Girardeau gas

What is a Tonks-Girardeau gas?

- ground state: fermionization (Girardeau 1960)

$$\psi_B(x_1,\dots,x_N) = \prod_{i<j}\left|\sin[\frac{\pi}{L}(x_i-x_j)]\right| = |\psi_F(x_1,\dots,x_N)|$$

Plot of $\psi_B(x, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$:



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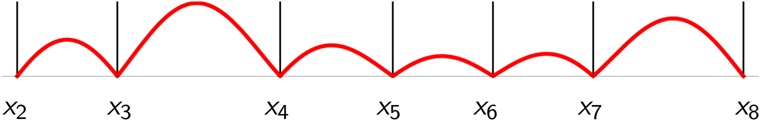


$\Longrightarrow$ strongly correlated system

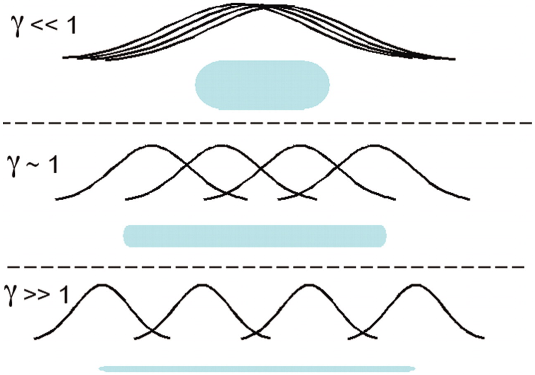
- $\mu_{TG} = \dfrac{\pi^2 \hbar^2 n^2}{2M}$: Fermi energy E_F for $k_F = \pi \frac{N}{L}$!
  independent of γ and thus of $\omega_\perp$
  By contrast, $\mu_{TF} = g_1 n \propto \omega_\perp$ at fixed n.

Experimental evidence for a Tonks-Girardeau gas

Kinoshita et al., 2004
Basic idea:

transverse squeezing changes axial expansion

transverse squeezing has no effect on the axial distribution (or energy)



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- trapped gas: local density approximation, valid for $\mu_0 \gg \hbar\omega_x$

$$\mu(n(x)) + U(x) = \mu_0 = N\hbar\omega_x = E_F \qquad \gamma(x) = \gamma(n(x))$$

Thomas-Fermi profile $n(x) = n_0\sqrt{1 - \frac{x^2}{R^2}}$ with $R = \sqrt{2N}\ell_x$

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- Plot the released longitudinal energy $\mu \gg \hbar\omega_\perp$ as a function of U_0 or $\gamma_{avg} = \langle\gamma(x)\rangle$

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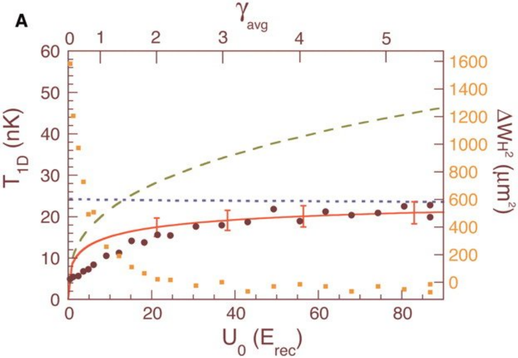
Results: released longitudinal energy $\mu \gg \hbar\omega_\perp$ measured by TOF



$$\mu_{\mathrm{TF}} = g_1 n \propto \omega_\perp \propto \sqrt{U_0}$$

$$\mu_{\mathrm{TG}} = \frac{\pi^2\hbar^2 n^2}{2M}$$

The released energy tends to the Tonks-Girardeau chemical potential, with all energy being kinetic.

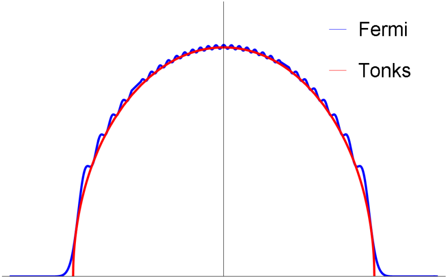
Are bosons just like fermions?

Bosons in the Tonks-Girardeau regime behave like fermions...

- same density distribution

Fermi: $\sum |\phi_n(x)|^2$
h.o. eigenfunctions

Tonks: TF distribution
$n(x) = n_0\sqrt{1 - \frac{x^2}{R^2}}$
= Fermi distribution for
$N \to \infty$



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$g^{(3)}(0) \propto \dfrac{n^3}{\gamma^6} \Longrightarrow$ 3-body losses strongly suppressed at large γ

Experiment by Laburthe Tolra et al., 2004: comparison of 3-body decay between a 3D BEC and a series of 1D tubes.

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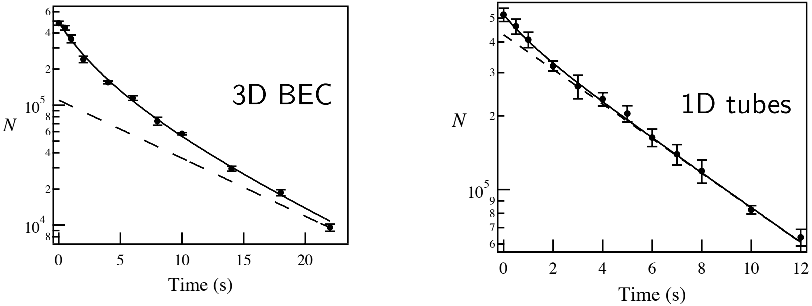

$\gamma \approx 0.5 \quad \frac{K_3^{1D}}{K_3^{3D}} = 0.14$ 3-body loss reduction by a factor 7!

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- consequence: very different momentum distribution

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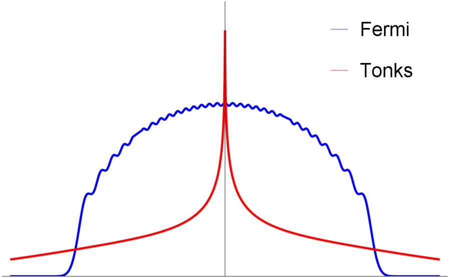
Fermi: $\sum |\phi_n(p)|^2$
h.o. eigenfunctions

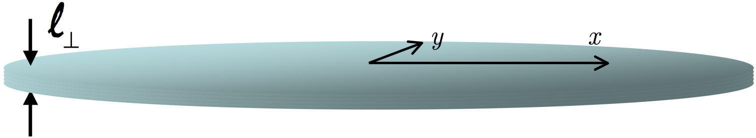
Tonks:
large p: $n(p) \propto \frac{1}{p^4}$
small p: $n(p) \propto \frac{1}{\sqrt{p}}$

see also: Paredes et al. (2004)



2D Bose gas

2D interacting Bose gas



Reminder: BEC or not BEC?

Reminder for a non interacting gas:

- uniform gas: $\varepsilon = \frac{\hbar^2 k^2}{2M}$ $\quad \mathbf{k} = \frac{2\pi}{L}(n_x, n_y)$ $\quad n(\varepsilon) = \frac{1}{\exp(\beta(\varepsilon - \mu)) - 1}$

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$\mu = 0$ only at $T = 0$.

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- trapped gas (harmonic trap): $N_C = \frac{\pi^2}{6}\left(\frac{k_B T}{\hbar\omega}\right)^2$ BEC

local density: like the uniform system, where μ is replaced by a local chemical potential $\mu - V(\mathbf{r})$.

$n(\mathbf{r})\lambda^2 = -\ln\left(1 - e^{\beta(\mu - V(\mathbf{r}))}\right)$ Below T_C, $\mu = 0$.

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But: at the centre $\mu = V(\mathbf{0}) = 0 \Longrightarrow n(\mathbf{0}) = \infty$
What will happen with interactions limiting the density?

Reminder: interacting 2D gas

interacting 2D gas:

- if $a > \ell_\perp$, $g_{2D} = \dfrac{4\pi\hbar^2}{M}\dfrac{1}{\ln(1/na_{2D}^2)}$ (strictly 2D)

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Typical value in the experiments: $\tilde{g}_2 = 0.13$ (ENS) to 0.02 (NIST) with $a \ll \ell_\perp \Longrightarrow$ weak interactions

The uniform interacting 2D gas

Phase coherence / long range order: $g^{(1)}(\mathbf{r}) \to g^{(1)}(\infty) \neq 0$?

The uniform interacting 2D gas

Phase coherence / long range order: $g^{(1)}(\mathbf{r}) \to g^{(1)}(\infty) \neq 0$?

- $T = 0$ Yes!
  - ideal gas: $\mu = 0$, all in ground state $\Rightarrow$ long range order
  - interacting gas: OK if weak interactions, *i.e.* gas parameter $\frac{1}{\ln(1/na_{2D}^2)} \ll 1$ (Schick 1971)

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- $T = 0$ Yes!

  - ideal gas: $\mu = 0$, all in ground state $\Rightarrow$ long range order

  - interacting gas: OK if weak interactions, *i.e.* gas parameter $\frac{1}{\ln(1/na_{2D}^2)} \ll 1$ (Schick 1971)

- $T > 0$ No!

  thermal fluctuations (phonons) destroy phase coherence (Hohenberg, Mermin, Wagner, 1966/1967)

  - ideal gas: $g^{(1)}(r) \propto e^{-r/\ell}$ exponential decay

  - interacting gas: phase fluctuations diverge logarithmically at infinity $\langle \delta\hat{\phi}^2 \rangle \sim \frac{2}{n_0\lambda^2} \ln(r/\xi)$ ξ: healing length

$$\Longrightarrow g^{(1)}(r) \sim \left(\frac{\xi}{r}\right)^{\frac{1}{n_0\lambda^2}} \quad \text{algebraic decay}$$

Superfluid transition

Above a critical temperature $T_C \sim T_d/10$, $g^{(1)}$ decays exponentially. Below T_C, $g^{(1)}$ decays algebraically. A fraction of the gas is superfluid, with a jump in superfluid density n_s between 0 and $n_S\lambda^2 = 4$

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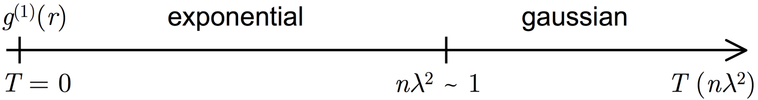

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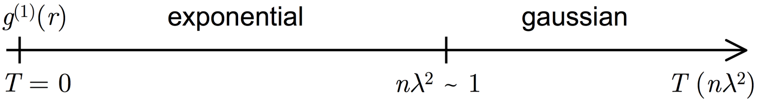

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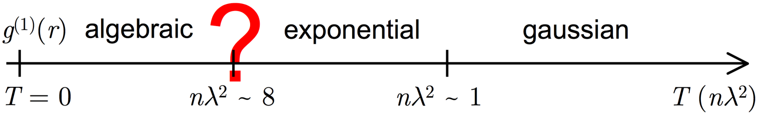

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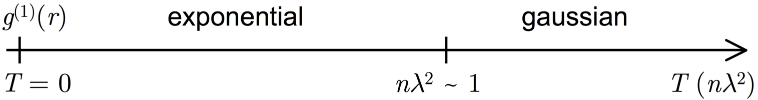

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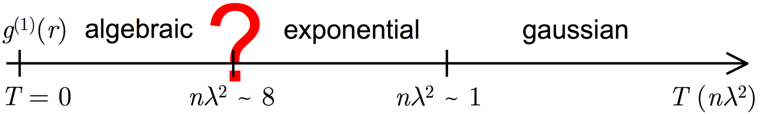

What is the nature of the transition?

Mechanisms for phase fluctuations

No BEC, but quasi-condensate with fluctuating phase and negligible density fluctuations.

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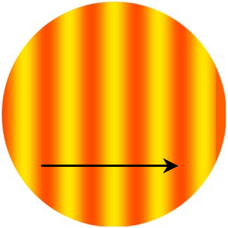

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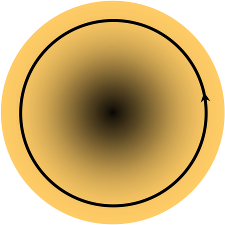
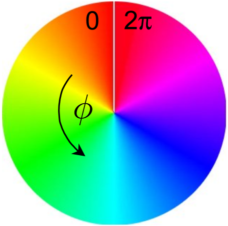


density hole of radius ξ



2π phase loop

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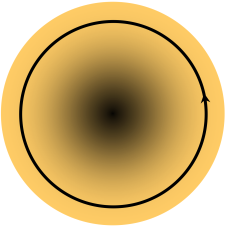

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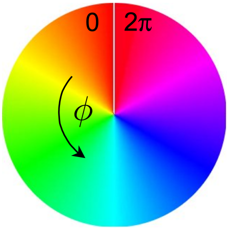

2π phase loop

Size of the core: ξ such that $\mu_{2D} = \dfrac{\hbar^2}{2M\xi^2}$.

Vortex nucleation

$\psi = \sqrt{n}e^{i\phi} \simeq \sqrt{n_0}e^{i\phi}$ fluctuates randomly.
Continuity of $\psi \Rightarrow$ continuity of Re ψ and Im ψ
$\Rightarrow$ lines of Re $\psi = 0$ and Im $\psi = 0$, which fluctuate.

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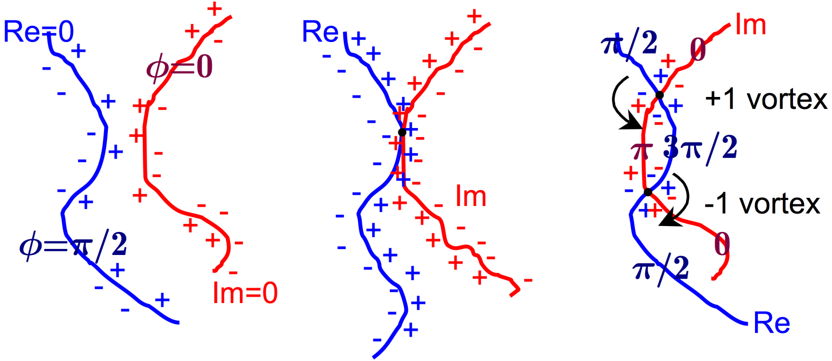
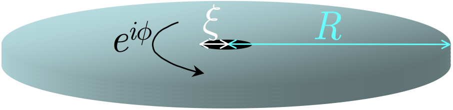


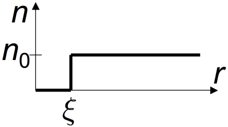
When two lines cross, 2 vortices of opposite charge are created.

Energy of a vortex

- Single vortex at the centre of a circular condensate of size R



- Model for the condensate density: step function of range ξ



- velocity field: $\mathbf{v} = \frac{\hbar}{M}\nabla\phi$
- Circulation of the velocity: $\oint \mathbf{v}\cdot d\mathbf{r} = q\,2\pi\frac{\hbar}{M}, \quad q \in \mathbb{Z}$
- Velocity field for a vortex of charge q: $\mathbf{v} = q\dfrac{\hbar}{Mr}\mathbf{u}_\phi$

Energy of a vortex

- Kinetic energy: $E_K = \int \frac{1}{2} M v^2(\mathbf{r}) n(\mathbf{r}) d\mathbf{r}$

$$E_K = \frac{1}{2} M n_0 \int_{\xi}^{R} 2\pi r\, dr\, q^2 \frac{\hbar^2}{M^2 r^2} = q^2 \frac{\hbar^2 n_0 \pi}{M} \ln \frac{R}{\xi}$$

$$E_K = q^2 n_0 \lambda^2 \frac{k_B T}{2} \ln \frac{R}{\xi}$$

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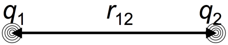
- Two vortices of charge q_1 and q_2:



$$E = E_K + E_I = n_0 \lambda^2 \left[(q_1 + q_2)^2 \ln\left(\frac{R}{\xi}\right) - 2 q_1 q_2 \ln\left(\frac{r_{12}}{\xi}\right) \right] \frac{k_B T}{2}$$

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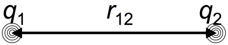

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- Several vortices of charge ± 1 more stable than multiply charged vortex.

The BKT transition

Berezinskii-Kosterlitz-Thouless transition by simple energetic arguments:

- Free energy $F = E - TS$:
  $F > 0$, low probability of vortex formation;
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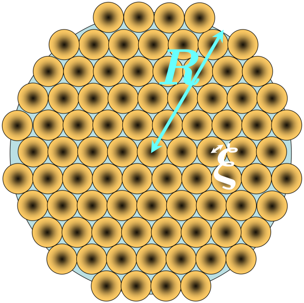

$$W = \frac{\pi R^2}{\pi \xi^2} \Longrightarrow S = 2 k_B \ln \frac{R}{\xi}$$

$$\text{or } TS = 4 \frac{k_B T}{2} \ln \frac{R}{\xi}$$

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- $F = \dfrac{k_B T}{2}\left(q^2 n_0 \lambda^2 - 4\right) \ln \dfrac{R}{\xi}$; easiest case $q = \pm 1$:

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Free energy $F = \frac{k_B T}{2} \left(n_0 \lambda^2 - 4 \right) \ln \frac{R}{\xi}$

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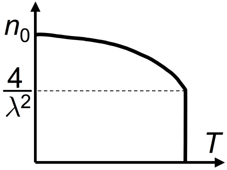
- $n_0 \lambda^2 < 4$ (high T): proliferation of free vortices; no superfluidity $\Rightarrow n_0 = 0$



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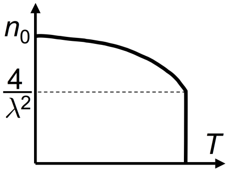

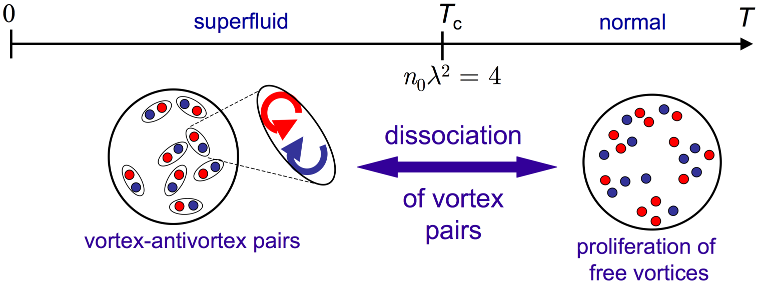

Hadzibabic et al. (2006)

Critical temperature

At T_C, n_0 jumps from 0 to $n_0\lambda^2 = 4$. What is $n_{\rm tot}$?
Prokof'ev et al. (2001):

$$\text{at } T_C: \qquad n_{\rm tot} = \ln\frac{C}{\tilde{g}_2}, \qquad C = 380$$

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$$\tilde{g}_2 = \sqrt{8\pi}\frac{a}{\ell_\perp} = ...? \qquad a = 2-5\,\text{nm} \qquad \ell_\perp = 40-100\,\text{nm}$$

Paris experiment: $\tilde{g}_2 = 0.13$ $\quad n_{\rm tot}\lambda^2 = 8$ at T_C
NIST experiment: $\tilde{g}_2 = 0.02$ $\quad n_{\rm tot}\lambda^2 = 10$ at T_C

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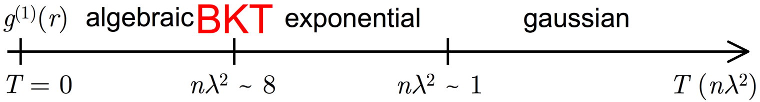

Finite size effects

What does change in a trap? Condensate fraction in the superfluid phase?

Recall: $g^{(1)}(r) \sim \left(\frac{\xi}{r}\right)^{\frac{1}{n_0\lambda^2}} \underset{r \to \infty}{\longrightarrow} 0$ no long range order.

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finite system L, $n_0\lambda^2 > 4$, $g^{(1)} \gtrsim \left(\frac{\xi}{L}\right)^{\frac{1}{4}}$

$\xi \sim 0.1\,\mu\text{m}$, $L \sim 100\,\mu\text{m} \Rightarrow g^{(1)}(L) \sim 0.2$ not so small!

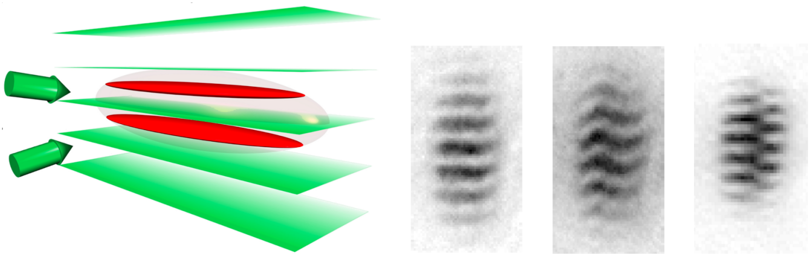
- Large condensate fraction f_0
- Large contrast in interference experiments

Achtung! $f_0 \neq \dfrac{n_0}{n}$

$$\begin{cases} f_0 : \text{occupation of the most populated single particle level} \\ n_0 : \text{superfluid density} \end{cases}$$

Experimental evidence for BKT in a 2D gas

ENS experiment: measure $g^{(1)}$ decay by interferometry



- measurement of the integrated contrast:

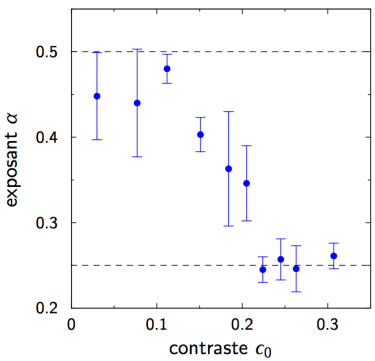
$$\frac{1}{L}\int_{-L}^{L} |g^{(1)}(x)|^2 \, dx \propto \frac{1}{L^{2\alpha}}$$

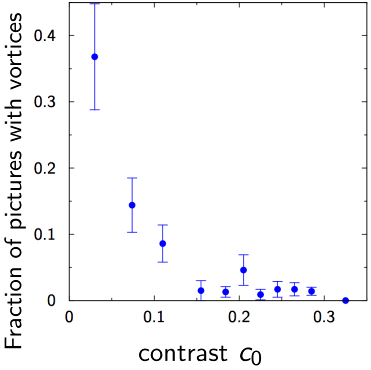
exponential decay: $\alpha = \frac{1}{2}$ / algebraic decay: $\alpha = \frac{1}{4}$

- statistics on phase defects (free vortices)

Experimental evidence for BKT in a 2D gas

Results: Hadzibabic et al., 2006





- contrast c_0 is a measure of temperature
- BKT transition evidenced by a step in exponent α and apparition of vortices

2D physics in ultracold gases

Experimental results:

- BKT identified via 1-body correlation

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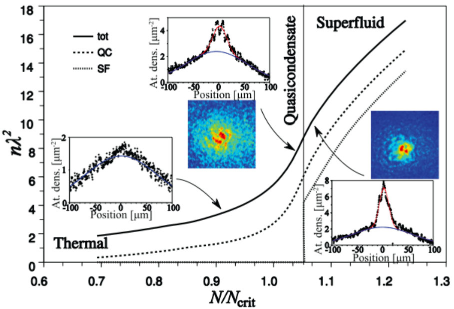
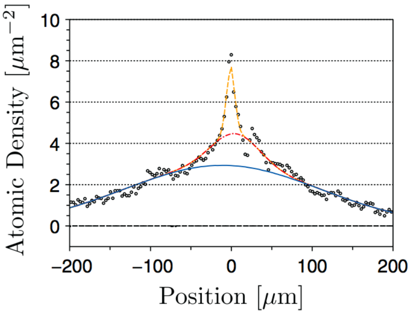
Density profiles at NIST

Cladé et al., 2008





Evidence for 3 phases (superfluid, QC and thermal)

$\neq$ ENS experiment (2 components only)

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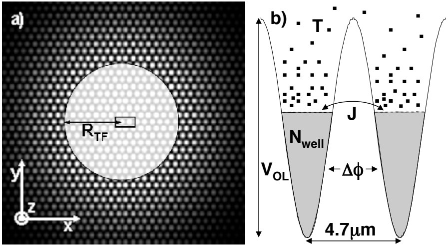
2D lattice of period 5 μm
temperature T
tunnel coupling J
thermal phase fluctuations
$\Delta\phi \approx \sqrt{T/J}$



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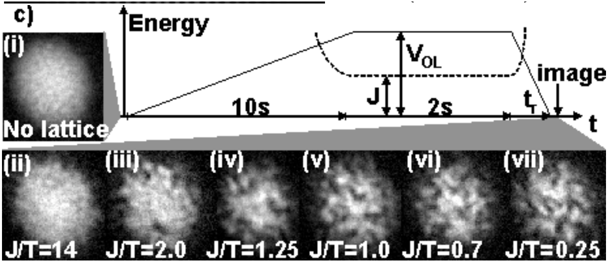
$\Delta\phi \approx \sqrt{T/J}$
proliferation of vortices for $J < T$



slowly remove the lattice to reconnect the phase before imaging

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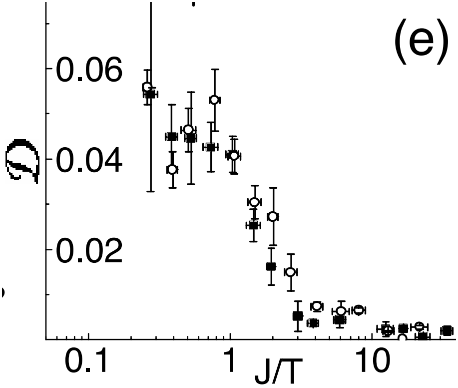
density of vortices $\mathcal{D}$:
results at two temperatures (35 nK and 60 nK) collapse on a single curve $\mathcal{D}\left(\frac{J}{T}\right)$

Schweikhard et al., 2007



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incomplete to do list:

- clarify the nature of the phases
- observe the vortex binding/unbinding
- frequency shift predicted for the 2ω Pitaevskii mode
- confinement induced scattering resonance
- FQHE in a rotating 2D gas
- ...