

## Light forces — Answers

### 1 Calculation of the mean force

#### 1.1 System under consideration and useful approximations

1. A photon momentum  $\hbar k_L$  is exchanged during absorption or emission of a single photon. The corresponding velocity change is  $\Delta v = v_{\text{rec}} = \hbar k_L/m$ . The kinetic energy of an atom moving at a velocity  $v_{\text{rec}}$  is

$$E_{\text{rec}} = \frac{1}{2} m v_{\text{rec}}^2 = \frac{\hbar^2 k_L^2}{2m}.$$

The recoil frequency is defined by  $\omega_{\text{rec}} = E_{\text{rec}}/\hbar = \hbar k_L^2/(2m)$ . The numbers for rubidium on the  $D2$  line are  $v_{\text{rec}} = 5.9 \text{ mm}\cdot\text{s}^{-1}$  and  $\omega_{\text{rec}} = 2\pi \times 3.8 \text{ kHz}$ .

The typical acceleration due to absorption of the laser photons (velocity transfers  $v_{\text{rec}}$ , all directed along  $\mathbf{k}_L$ ) followed by random spontaneous emission (with equal probability in opposite directions, and zero average contribution) at a maximum scattering rate  $\Gamma/2$  set by the natural line width is  $\Gamma v_{\text{rec}}/2$ . This corresponds to about  $10^5 \text{ m}\cdot\text{s}^{-2}$ , that is ten thousand times the Earth acceleration! We expect that the light forces will be very efficient to manipulate the external degrees of freedom of the atoms.

**Remark** Strictly speaking, the recoil energy must be taken into account in the resonance condition. Let us consider a transition between  $g$  and  $e$  induced by a laser at a frequency  $\omega$ . The initial energy if  $N$  photons are present initially in the laser field is

$$E_i = 0 + N\hbar\omega + \frac{1}{2}mv^2.$$

The final momentum is  $m\mathbf{v} + \hbar\mathbf{k}_L$ , so that the final energy is

$$E_f = \hbar\omega_0 + (N-1)\hbar\omega + \frac{1}{2m} (m\mathbf{v} + \hbar\mathbf{k}_L)^2 = \hbar\omega_0 + (N-1)\hbar\omega + \frac{1}{2}mv^2 + \hbar\mathbf{k}_L \cdot \mathbf{v} + E_{\text{rec}}.$$

The resonance condition thus writes:

$$E_i = E_f \Leftrightarrow \omega - \mathbf{k}_L \cdot \mathbf{v} = \omega_0 + \omega_{\text{rec}}.$$

The Doppler shifted laser frequency should hence be equal to the atomic frequency plus the recoil frequency. We have seen that, at least for rubidium (and this is true for all D lines of alkali), the recoil frequency is much less than the linewidth  $\Gamma$  of the transition, which makes this  $E_{\text{rec}}$  correction irrelevant. However, it becomes important for narrow lines, and if a two-photon Raman transition is used.

2. The atomic Hamiltonian is

$$\hat{H}_A = \hbar\omega_0 |e\rangle\langle e| + \frac{\hat{\mathbf{P}}^2}{2m}.$$

The light is described by a classical field. The atom-laser coupling term in the dipolar representation is

$$\begin{aligned}\hat{V}_{AL} &= -\hat{\mathbf{D}} \cdot \mathbf{E}_L(\hat{\mathbf{R}}, t) = \hat{V}_{AL}^{\text{res}} + \hat{V}_{AL}^{\text{non res}} \\ \text{where } \hat{V}_{AL}^{\text{res}} &= -\frac{1}{2} \left( \mathbf{d} \cdot \boldsymbol{\epsilon}(\hat{\mathbf{R}}) \right) \mathcal{E}_L(\hat{\mathbf{R}}) |e\rangle\langle g| e^{-i\omega t - i\phi(\hat{\mathbf{R}})} + h.c. \\ \text{and } \hat{V}_{AL}^{\text{non res}} &= -\frac{1}{2} \left( \mathbf{d} \cdot \boldsymbol{\epsilon}^*(\hat{\mathbf{R}}) \right) \mathcal{E}_L^*(\hat{\mathbf{R}}) |e\rangle\langle g| e^{i\omega t + i\phi(\hat{\mathbf{R}})} + h.c.\end{aligned}\quad (1)$$

In the interaction picture with respect to the free Hamiltonian  $\hat{H}_A$ , the operator  $|e\rangle\langle g|$  (respectively  $|g\rangle\langle e|$ ) evolves as  $|e\rangle\langle g|(t=0) e^{i\omega_0 t}$  (respectively  $|g\rangle\langle e|(t=0) e^{-i\omega_0 t}$ ). In this picture,  $\hat{V}_{AL}^{\text{res}}$  evolves thus only at a rate  $\delta$  whereas  $\hat{V}_{AL}^{\text{nonres}}$  evolves at a much higher rate  $\omega + \omega_0$ . The rotating wave approximation consists in neglecting this non resonant last term and keeping only  $\hat{V}_{AL}^{\text{res}}$ . We finally have:

$$\hat{H} = \hbar\omega_0 |e\rangle\langle e| + \frac{\hat{\mathbf{P}}^2}{2m} + \frac{\hbar\Omega_1(\hat{\mathbf{R}})}{2} |e\rangle\langle g| e^{-i\omega t - i\phi(\hat{\mathbf{R}})} + h.c. \quad (2)$$

where we used the expression of the Rabi frequency, defined through  $\hbar\Omega_1(\mathbf{r}) = -(\mathbf{d} \cdot \boldsymbol{\epsilon}(\mathbf{r})) \mathcal{E}_L(\mathbf{r})$ .

3. The typical time scale for the evolution of the internal variables is set by  $T_{\text{int}} \simeq \Gamma^{-1}$ . The time scale for the external variables can be estimated as the time needed for an atom accelerated by the laser and initially on resonance to get out of resonance due to the Doppler shift. Under the maximal acceleration  $k_L \Gamma v_{\text{rec}}/2$  due to spontaneous scattering of the laser photons, this time is set by  $k_L \Gamma v_{\text{rec}} T_{\text{ext}}/2 = \Gamma$ . We thus have

$$T_{\text{ext}} \simeq \frac{2}{k_L v_{\text{rec}}} = \frac{2m}{\hbar k_L^2} = \omega_{\text{rec}}^{-1}.$$

This is known as the recoil time  $t_{\text{rec}} = \omega_{\text{rec}}^{-1}$ , and is in general very large as compared to  $T_{\text{int}}$ . For rubidium for example, the recoil frequency is 3.8 kHz, very small as compared to  $\Gamma/(2\pi) = 6$  MHz. On the time scale of the external motion, the internal variables always have the time to reach their steady state, so that when we study the external motion, we can assume that the internal variables adiabatically follow their steady state.

4. The spread of the atomic wave packet  $\Delta R$  must be much smaller than the scale  $\lambda$  (or, say,  $k_L^{-1}$  of the variations of the laser field. In addition, the momentum spread must be small enough for the frequency seen by the atom to be well defined for the given transition: the dispersion in the Doppler shift must be much smaller than  $\Gamma$ . We are left with the two conditions:

$$\Delta R \ll \frac{1}{k_L} \quad k_L \frac{\Delta P}{m} \ll \Gamma.$$

As the Heisenberg uncertainty still holds, this implies

$$\frac{\hbar}{2} < \Delta R \Delta P \ll \frac{\Gamma m}{k_L^2} \Leftrightarrow \omega_{\text{rec}} \ll \Gamma.$$

We recover the same condition which is necessary to well separate the time scales of the internal and external variables! This condition, which stipulates that the line width must be larger than the recoil frequency, is known as the *broadband condition*. It is well satisfied for the strong lines usually used in laser cooling, in particular for the  $D1$  or  $D2$  lines of alkali. Under this condition, we are allowed to treat the external motion of the atom classically and replace operators by their average values.

## 1.2 Calculation of the mean force

1. The equation of motion for the position and momentum operators in the Heisenberg picture are as follows:

$$\frac{d\hat{\mathbf{R}}}{dt} = \frac{1}{i\hbar} [\hat{\mathbf{R}}, \hat{H}], \quad (3)$$

$$\frac{d\hat{\mathbf{P}}}{dt} = \frac{1}{i\hbar} [\hat{\mathbf{P}}, \hat{H}]. \quad (4)$$

The term  $\hbar\omega_0 |e\rangle\langle e|$  in  $\hat{H}$  only affects the internal variables, and commutes with both  $\hat{\mathbf{R}}$  and  $\hat{\mathbf{P}}$ . The kinetic energy commutes with  $\hat{\mathbf{P}}$ , but not with  $\hat{\mathbf{R}}$  and we have

$$\frac{d\hat{\mathbf{R}}}{dt} = \frac{\hat{\mathbf{P}}}{m},$$

which links the atomic velocity with momentum. The derivative of the momentum operator is the *force operator* and arises from the commutator between  $\hat{\mathbf{P}}$  and  $\hat{V}_{AL}^{\text{res}}$ , which contains terms in  $\hat{\mathbf{R}}$ :

$$\hat{\mathbf{F}} = \frac{d\hat{\mathbf{P}}}{dt} = -\nabla \hat{V}_{AL}^{\text{res}} = -\nabla \left\{ \frac{\hbar\Omega_1(\hat{\mathbf{R}})}{2} |e\rangle\langle g| e^{-i\omega t - i\phi(\hat{\mathbf{R}})} + h.c. \right\}$$

The mean force is the average of the force operator:

$$\mathbf{F} = \langle \hat{\mathbf{F}} \rangle = - \left\langle \nabla \left( \frac{\hbar\Omega_1(\hat{\mathbf{R}})}{2} |e\rangle\langle g| e^{-i\omega t - i\phi(\hat{\mathbf{R}})} \right) \right\rangle + c.c.$$

We make the semi-classical approximation and replace the position operator by its average value. Noticing that the addition of the complex conjugate will give twice the real part of the first term, we have:

$$\mathbf{F} = -\Re \left\{ (\hbar\nabla\Omega_1 - i\hbar\Omega_1\nabla\phi) \langle |e\rangle\langle g| e^{-i\omega t - i\phi(\mathbf{r})} \rangle \right\}$$

The average is only on the internal variables now. We already remark that the force is the sum of two terms, the first one being proportional to the gradient in Rabi frequency and the other one to the phase gradient.

2. We discussed earlier that the difference in time scales allows us to consider that the internal variables always follow their steady state on the time scale of the external motion. We can then replace the average at time  $t$  by the average taken in the steady state:

$$\mathbf{F} = -\Re \left\{ (\hbar\nabla\Omega_1 - i\hbar\Omega_1\nabla\phi) \left\langle |e\rangle\langle g| e^{-i\omega t - i\phi(\mathbf{r})} \right\rangle_{\text{st}} \right\}. \quad (5)$$

3. Let us explicit the average value:

$$\left\langle |e\rangle\langle g| e^{-i\omega t - i\phi(\mathbf{r})} \right\rangle_{\text{st}} = \text{Tr} \left( \hat{\rho}_{\text{st}} |e\rangle\langle g| e^{-i\omega t - i\phi(\mathbf{r})} \right) = (\tilde{\rho}_{ge})_{\text{st}} = u_{\text{st}} + iv_{\text{st}}.$$

The mean force will then finally read as a function of  $u$  and  $v$ :

$$\mathbf{F} = -u_{\text{st}} \hbar\nabla\Omega_1 - v_{\text{st}} \hbar\Omega_1\nabla\phi. \quad (6)$$

The optical Bloch equations will allow us to calculate the steady state value of  $\tilde{\rho}_{ge} = u + iv = \rho_{ge} e^{-i\omega t - i\phi(\mathbf{r})}$ . We have:

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} - i\frac{\Omega_1}{2}(\tilde{\rho}_{ge} - \tilde{\rho}_{ge}^*) = -\Gamma\rho_{ee} + \Omega_1 v, \quad (7)$$

$$\frac{d\tilde{\rho}_{ge}}{dt} = \left(-i\delta - \frac{\Gamma}{2}\right)\tilde{\rho}_{ge} + i\frac{\Omega_1}{2}(1 - 2\rho_{ee}), \quad (8)$$

where the right hand sides do not depend on  $e^{\pm i\omega t}$  anymore. The  $\mathbf{r}$  dependence has been omitted but is still there. In the steady state, the left hand sides vanish and we have

$$\begin{aligned} \rho_{ee} &= \frac{\Omega_1 v}{\Gamma}, \\ \tilde{\rho}_{ge} &= u + iv = \frac{\Omega_1/2}{\delta - i\Gamma/2}(1 - 2\rho_{ee}) = \frac{\Omega_1/2}{\delta^2 + \Gamma^2/4}(1 - 2\rho_{ee})\left(\delta + i\frac{\Gamma}{2}\right). \end{aligned}$$

4. Introducing the saturation parameter  $s$  and identifying the real and imaginary parts of  $\tilde{\rho}_{ge}$  with  $u$  and  $v$ , we get

$$u = \frac{2\delta}{\Gamma}v \quad \text{and} \quad v = s\left(\frac{\Gamma}{2\Omega_1} - \frac{\Gamma\rho_{ee}}{\Omega_1}\right) = s\frac{\Gamma}{2\Omega_1} - sv.$$

We end up with

$$v = \frac{s}{1+s}\frac{\Gamma}{2\Omega_1} \quad u = \frac{s}{1+s}\frac{\delta}{\Omega_1} \quad \rho_{ee} = \frac{1}{2}\frac{s}{1+s}.$$

5. The saturation intensity for the  $D2$  line of rubidium is  $16.7 \text{ W}\cdot\text{m}^{-2}$ , or equivalently  $1.67 \text{ mW}\cdot\text{cm}^{-2}$ . This intensity is easy to achieve with lasers of moderate power, not strongly focused. This is good news for the applications of light forces in the manipulation of the external motion of atoms.
6. Replacing the expression of  $u$  and  $v$  into (6), we finally obtain the expression of the mean force:

$$\mathbf{F} = -\frac{s(\mathbf{r})}{1+s(\mathbf{r})}\left(\hbar\delta\frac{\nabla\Omega_1}{\Omega_1(\mathbf{r})} + \frac{\hbar\Gamma}{2}\nabla\phi\right). \quad (9)$$

The first term, arising from  $u_{\text{st}}$ , comes from the component of the atomic dipole which oscillates in phase with the laser field. The associated force is the dipole force

$$\mathbf{F}_{\text{dip}} = -\frac{\hbar\delta}{2}\frac{s(\mathbf{r})}{1+s(\mathbf{r})}\frac{\nabla s}{s} = -\nabla U_{\text{dip}}.$$

It derives from a potential  $U_{\text{dip}}$  given by

$$U_{\text{dip}} = \frac{\hbar\delta}{2}\ln(1+s(\mathbf{r})). \quad (10)$$

It is thus a conservative force, associated with dispersion. The microscopic process responsible for this force is the absorption of a laser photon, followed by a stimulated emission into the laser field.

On the other hand, the other part of the force comes from the component of the dipole  $v_{\text{st}}$  oscillating in quadrature with the laser field, and is thus associated with absorption. It is called the radiation pressure and its expression is

$$\mathbf{F}_{\text{pr}} = \frac{\Gamma}{2}\frac{s(\mathbf{r})}{1+s(\mathbf{r})}(-\hbar\nabla\phi).$$

In order to understand better its effect, let us consider the radiation pressure associated to a plane wave, where the phase is  $\phi(\mathbf{r}) = -\mathbf{k}_L \cdot \mathbf{r}$ . The radiation pressure is in this case

$$\mathbf{F}_{\text{pr}} = \frac{\Gamma}{2} \frac{s(\mathbf{r})}{1 + s(\mathbf{r})} \hbar \mathbf{k}_L. \quad (11)$$

This is nothing but the spontaneous scattering rate  $\Gamma \rho_{ee} = \frac{\Gamma}{2} \frac{s}{1+s}$  multiplied by the recoil of a single photon. This force comes from absorptions of the laser photons followed by spontaneous emissions, as discussed in the beginning of these notes. This force is dissipative.

7. If the atoms move with a velocity  $\mathbf{v}$ , the detuning  $\delta$  must be replaced by  $\delta - \mathbf{k}_L \cdot \mathbf{v}$ . The value of  $s$  must be changed accordingly, so that  $s$  is now a function of  $\mathbf{r}$  and  $\mathbf{v}$ .
8. The ratio  $F_{\text{dip}}/F_{\text{pr}}$  is of order

$$\frac{F_{\text{dip}}}{F_{\text{pr}}} \simeq \frac{|\delta|}{\Gamma} \frac{1}{k_L} \frac{|\nabla s|}{s}.$$

The saturation parameter depends on position because of intensity or polarization gradients, which vary at most on the scale  $k_L^{-1}$ , in a standing wave.  $F_{\text{dip}}/F_{\text{pr}}$  is thus less than  $|\delta|/\Gamma$ . Near resonance, the radiation pressure dominates over the dipole force. However, for a detuning very large as compared to the line width, the dipole force decreases only as  $1/\delta$  and can be made dominant in front of the radiation pressure, which decays like  $1/\delta^2$ . In this situation, the force is essentially conservative, and light forces can be used to create conservative traps for ultra cold atoms, including Bose-Einstein condensates [1].

## 2 Applications of the light forces

### 2.1 Zeeman slower

1. The Rabi frequency is homogeneous for a plane wave, which means that the dipole force vanishes. For a plane wave of wave vector  $\mathbf{k}_L$ ,  $\phi(\mathbf{r}) = -\mathbf{k}_L \cdot \mathbf{r}$ . The radiation pressure is thus given by Eq.(11) above. If the atom is moving with velocity  $\mathbf{v}$ , its expression is

$$\mathbf{F}_{\text{pr}} = \frac{\Gamma}{2} \frac{\Omega_1^2/2}{\Omega_1^2/2 + (\delta - \mathbf{k}_L \cdot \mathbf{v})^2 + \Gamma^2/4} \hbar \mathbf{k}_L.$$

2. For a given initial velocity  $\mathbf{v}_0$ , the deceleration due to a counter propagating plane wave will be maximum if the detuning is chosen to be  $\delta = \mathbf{k}_L \cdot \mathbf{v}_0$ . However, as the atoms are decelerated, their velocity change and they get out of resonance as soon as it has changed by a few times  $\Gamma/k_L$  (about  $5 \text{ m}\cdot\text{s}^{-1}$  for rubidium). The deceleration will thus only lower the velocity by a few tens of meters per second.
3. The idea of the Zeeman slower is to tailor the magnetic field profile to compensate for the Doppler shift at each position  $z$ . We look for a profile such that

$$\omega - \mathbf{k}_L \cdot \mathbf{v}(z) = \omega'_0(z) = \omega_0 + \gamma B(z),$$

or in other words, if  $v(z) > 0$  is the projection of the velocity on the  $z$  axis and the laser propagates towards the  $z < 0$  direction,

$$\gamma B(z) = \delta + k_L v(z).$$

If this condition is fulfilled, the light is on resonance at every point, and, provided that the laser intensity is large as compared to the saturation intensity, the atoms are decelerated with a constant, maximal acceleration  $a = -\hbar k_L \Gamma / (2m) = -\Gamma v_{\text{rec}} / 2$ . The constant force is associated to the potential  $-Fz = m\Gamma v_{\text{rec}} z / 2$ . The atomic velocity at position  $z$  can be deduced from the conservation of total energy:

$$\frac{1}{2} m v_0^2 = m \frac{\Gamma}{2} v_{\text{rec}} L_0 = \frac{1}{2} m v(z)^2 + m \frac{\Gamma}{2} v_{\text{rec}} z,$$

where  $L_0 = v_0^2 / (\Gamma v_{\text{rec}})$  is the total length of the Zeeman slower, which leads to

$$v(z) = \sqrt{v_0^2 - \Gamma v_{\text{rec}} z} = v_0 \sqrt{1 - \frac{z}{L_0}}.$$

The profile of the magnetic field has to be a square root:

$$B(z) = B_0 + \Delta B \sqrt{1 - \frac{z}{L_0}}, \quad \text{with} \quad B_0 = \frac{\delta}{\gamma} \quad \text{and} \quad \Delta B = \frac{k_L v_0}{\gamma}.$$

N.B.  $B(z)$  should not change its sign, to prevent spin flips as  $B$  vanishes. As  $\Delta B > 0$ ,  $B_0$  should also be positive. Choosing  $B_0 = 0$  would mean that the laser is resonant with atoms at rest, which must be avoided in order to avoid re-excitation of the slowed atoms. It is thus better to chose  $\delta$  significantly larger than  $\Gamma$ .

N.B. If another transition is used, and thus another laser polarization, such that  $\omega'_0(z) = \omega_0 - \gamma B(z)$ , a detuning close to  $\delta = -k_L v_0$  can be used, and the magnetic field then increases from almost zero to  $B_0 = k_L v_0 / \gamma$  when  $z$  increases from 0 to  $L_0$  (we take  $\Delta B = -B_0$ ). In this configuration, the laser is naturally far detuned from the atomic transition of the slowed atoms at the output of the Zeeman slower.

4. Numerical estimate: for  $v_0 = 1000 \text{ m}\cdot\text{s}^{-1}$  and sodium atoms:

$$m = 3.82 \times 10^{-26} \text{ kg}, \quad \Gamma = 61.5 \times 10^6 \text{ s}^{-1}, \quad \lambda_0 = 589 \text{ nm}, \quad \gamma / (2\pi) = 14 \text{ GHz} \cdot \text{T}^{-1}.$$

We have  $v_{\text{rec}} = \hbar k_L / m = 29.4 \text{ mm}\cdot\text{s}^{-1}$ , and

$$L_0 = \frac{v_0^2}{\Gamma v_{\text{rec}}} = 0.55 \text{ m}.$$

This length is in agreement with Phillips' experiment, where the length was 60 cm. The maximum magnetic field is of order

$$\Delta B = \frac{k_L v_0}{\gamma} = 0.12 \text{ T} = 1200 \text{ G}.$$

This magnetic field is relatively high but can be produced with regular copper coils wound around a vacuum tube.

## 2.2 Optical lattice

We now consider the situation where two counter-propagating laser beams with identical linear polarization and same intensity create a standing wave. This situation is known as an *optical lattice* [2].

1. We have seen in section 1.2, question 8, that the dipole force will dominate over the radiation pressure for a large detuning:  $|\delta| \gg \Gamma$ . For rubidium, this means that the detuning should be larger than, say,  $10\Gamma$ , that is  $|\delta|/(2\pi) > 60$  MHz. In reality, the rubidium atom has an hyperfine structure in the excited state. To be able to re-sum the contribution of all the excited states, the detuning should be much larger than the width of the hyperfine structure, which is about 400 MHz. A detuning of several GHz is thus preferable.

N.B. If the detuning is even larger than the fine structure, the  $D1$  and  $D2$  lines at 795 nm and 780 nm contributing with approximately the same detuning, we are left with a  $L = 0 \rightarrow L = 1$  transition, where all Zeeman sublevels of the ground state are shifted equally, whatever the polarization of the laser, which may be interesting for some applications. This requires however a laser with a wavelength larger than 985 nm or lower than 655 nm for a detuning larger than the fine structure by a factor 10 at least. The laser intensity should then be very large for reaching a similar trap depth.

2. We have seen at Eq.(10) that

$$U_{\text{dip}}(\mathbf{r}) = \frac{\hbar\delta}{2} \ln(1 + s(\mathbf{r})).$$

For a large detuning, the saturation parameter is

$$s(\mathbf{r}) = \frac{\Omega_1(\mathbf{r})^2/2}{\delta^2 + \Gamma^2/4} \simeq \frac{\Omega_1(\mathbf{r})^2}{2\delta^2}.$$

If, in addition<sup>1</sup>,  $\Omega_1^2 \ll \delta^2$ , we can take the limit  $s \ll 1$  in the expression of the dipole potential:

$$U_{\text{dip}}(\mathbf{r}) \simeq \frac{\hbar\delta}{2} s(\mathbf{r}) \simeq \frac{\hbar\Omega_1^2(\mathbf{r})}{4\delta}.$$

This expression is nothing but the light shift of the  $|g\rangle$  level in the perturbation theory, with a coupling  $\hbar\Omega_1/2$  and a dominant contribution of a single transition detuned by  $\delta$ .

3. The light field of a standing wave can be written as

$$E(z, t) = \frac{\mathcal{E}_L}{2} \left( e^{-i\omega t + ik_L z} - e^{-i\omega t - ik_L z} \right) + c.c. = i\mathcal{E}_L \sin(k_L z) e^{-i\omega t} + c.c.$$

We chose a phase difference  $\pi$  for simplicity. With respect to the Rabi frequency  $\Omega_0$  of a single plane wave, the Rabi frequency of the standing wave is now

$$\Omega_1(z) = 2i\Omega_0 \sin(k_L z).$$

The associated dipole potential is

$$U_{\text{dip}}(z) = U_0 \sin^2(k_L z) \quad \text{where} \quad U_0 = \frac{\hbar\Omega_0^2}{\delta} = \frac{\hbar\Gamma^2}{\delta} \frac{I_0}{2I_s}.$$

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<sup>1</sup>This condition is easily achieved: for a detuning of 6 GHz corresponding to  $\delta = 1000\Gamma$ , it means that the intensity should be

4. The dipole potential has minima for each  $z = p\frac{\lambda_L}{2}, p \in \mathbb{Z}$ . Close to  $z = 0$  for example, we can expand the potential to quadratic order and find:

$$U_{\text{dip}}(z) \simeq U_0 k_L^2 z^2 = \frac{1}{2} m \omega_z^2 z^2,$$

where

$$\omega_z = \sqrt{\frac{2U_0 k_L^2}{m}} = \frac{2\sqrt{U_0 E_{\text{rec}}}}{\hbar} = 2\sqrt{\frac{U_0}{E_{\text{rec}}}} \omega_{\text{rec}}. \quad (12)$$

The natural scale for the well depth in a lattice is again the recoil energy.

### 2.3 Optical clocks

1. The states  $|g, n_z\rangle$  and  $|e, n'_z\rangle$  have energies

$$\begin{aligned} E_{g, n_z} &= \left(n_z + \frac{1}{2}\right) \hbar \omega_z \\ E_{e, n'_z} &= \hbar \omega_0 + \left(n'_z + \frac{1}{2}\right) \hbar \omega_z \end{aligned}$$

We expect transitions between  $|g, n_z\rangle$  and  $|e, n'_z = n_z + \Delta n\rangle$  at frequencies

$$\omega_{\Delta n} = \omega_0 + \Delta n \omega_z \quad \text{where } \Delta n \in \mathbb{Z}.$$

2.  $\eta = k_L \Delta z$  is the Lamb Dicke parameter, with  $\Delta z = \sqrt{\frac{\hbar}{2m\omega_z}}$ . In the Lamb-Dicke regime,  $\eta \ll 1$ , which implies

$$\frac{\hbar k_L^2}{2m\omega_z} \ll 1 \Leftrightarrow \omega_z \gg \omega_{\text{rec}}.$$

For the harmonic condition to be valid, it should at least be true for the ground state of width  $\Delta z$ , and we must have  $k_L \Delta z \ll 1$ , which is the Lamb-Dicke condition.

3. The Lamb-Dicke condition is equivalent to  $\Delta p \gg \hbar k_L$ ,  $\Delta p = \sqrt{m\hbar\omega_z/2}$  being the momentum spread of the ground state in the well. Therefore, the recoil kick after each absorption or emission modifies very little the atomic external state. As the different trapping states are orthogonal, the spontaneous emission in the lattice occurs essentially without changing  $n_z$ .

We can write this more explicitly by using the creation and annihilation operators in the harmonic trap and replacing  $z$  by its quantized form  $\hat{z} = \Delta z (a_z + a_z^\dagger)$  in the coupling term. The operator allowing an external state change induced by another probe beam, say, a resonant plane wave  $e^{ik_p z}$  propagating along  $z$ , is

$$e^{ik_p \Delta z (a_z + a_z^\dagger)} \simeq e^{i\eta (a_z + a_z^\dagger)} \simeq 1 + i\eta (a_z + a_z^\dagger)$$

where we have used the Lamb-Dicke regime assumption and the fact that  $k_p \simeq k_L$ . The coupling between states  $|e, n_z\rangle$  and  $|g, n_z \pm 1\rangle$  is thus smaller by a factor  $\sqrt{n_z + 1}\eta$  or  $\sqrt{n_z}\eta$  than the coupling between  $|e, n_z\rangle$  and  $|g, n_z\rangle$ , giving a transition probability smaller by a factor  $(n_z + 1)\eta^2$  or  $n_z\eta^2$ . At first order, only the carrier and the transitions at frequencies  $\omega_0 \pm \omega_z$  (the blue and red sidebands) remain. The sidebands can be excited with a high power beam, but spontaneous emission back to  $g$  occurs mainly on the carrier, at constant  $n_z$ .

4. Atoms confined in an optical lattice at a wavelength such that the main transition is unaffected (magic wavelength) are free from Doppler effect: the sidebands at  $\omega_0 \pm \omega_z$  are well separated from the main line at  $\omega_0$ , which has no Doppler broadening. This property is used in modern lattice clocks [3].
5. On the spectrum, we see the three lines: the carrier and the two first sidebands. The oscillation frequency in the lattice is the separation between lines, which is about 210 kHz. This value is in agreement with the value 212 kHz calculated from Eq.(12), and is much larger than the recoil frequency of 3.46 kHz. The atoms are deep in the Lamb-Dicke regime.  
 The side bands are driven by an intense laser field. Their weight is very different, because of a small mean  $\bar{n}_z$  value of  $n_z$ : the ratio between the red and the blue side bands is approximately  $\bar{n}_z/(\bar{n}_z + 1) = \exp[-\hbar\omega_z/(k_B T)]$ , which becomes small at low temperatures. Here the oscillation frequency corresponds to a temperature of about 10  $\mu\text{K}$ , larger than the cloud temperature, and we expect  $\bar{n}_z/(\bar{n}_z + 1) = 3\%$ . The ground state  $n_z = 0$  is the most populated, and is not coupled through the red sideband (there is no  $n_z = -1$  state). This is why the spectrum is very asymmetric.
6. In order to generalize this scheme to three dimensions, we need three standing waves, with different frequencies to avoid interferences between standing waves. The frequency difference should be large enough for the time dependent interference to average on the time scale of atomic motion. For example, a 80 MHz frequency shift provided by an acousto-optic modulator device is a valid choice.
7. If all atoms were in the ground state, the red sideband would vanish completely.

## References

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