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Hélène Perrin - Julien Laurat

Quantum opto-mechanical devices

Recently, dramatic progress has been achieved on pushing macroscopic mechanical systems to the quantum regime [1, 2]. The idea is to use the back action of light onto a reflecting surface, the radiation pressure already studied in the last exercise class in the case of single atoms, in order to cool the mechanical system. In this exercise class, we will consider a system consisting of a Fabry-Perot cavity with one mirror attached to a mechanical oscillator.

1 Classical field in a Fabry-Perot cavity

The system under consideration is a linear Fabry-Perot cavity of length L_0, with two mirrors M_1 and M_2, aligned along the x axis, see Fig. 1.

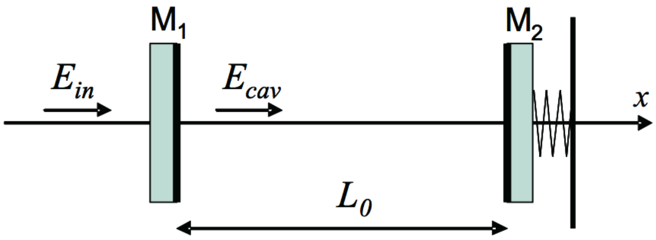

Figure 1: Optical cavity coupled to a mechanical oscillator. The fixed mirror M_1 is at position $x_1 = -L_0$, the mirror M_2 is at a mobile position x_2.

1.1 Fixed mirrors

In a first approach, we consider that both mirrors are fixed, spaced by L_0.

1. Give the eigenfrequencies ω_n of the intra-cavity field.

2. The mirror M_2 is supposed to reflect light perfectly ($r_2 = -1$), whereas M_1 has a reflection coefficient $r_1 = -1 + \gamma$ where $\gamma \ll 1$. The transmission coefficient t_1 is linked to r_1 through $r_1^2 + t_1^2 = 1$. Give its value at lowest order in γ. In the following, we denote $r = r_1 r_2 = 1 - \gamma$ for a round-trip.

3. A laser field $E_{\rm in}\, e^{-i\omega t + ik(x-x_1)} + c.c.$, modeled by a plane wave with complex amplitude $E_{\rm in}$, is send onto the M_1 mirror. Its angular frequency is ω, close to a resonance $\omega_0 = \omega_{n_0}$ of the cavity. The detuning is denoted as $\Delta = \omega - \omega_0$, and the wave vector is $k = \omega/c$. What is the steady state component $E_{\rm cav}^+$ of the electric field in the cavity, propagating along $+x$, at lowest order in γ and Δ? Deduce the following relation for the intra-cavity total mean intensity $I_{\rm cav} = 2I_{\rm cav}^+ = 2 \times 2\varepsilon_0\, c|E_{\rm cav}^+|^2$:
$$\frac{I_{\rm cav}}{I_{\rm in}} = \frac{A}{1 + 4\Delta^2/\Gamma^2}. \tag{1}$$
Give the expression and the physical meaning of A and Γ.

4. From a momentum conservation argument, show that the mean radiation pressure F of the field exerted onto the mirror M_2 reads:
$$F = \frac{S I_{\rm cav}}{c}, \tag{2}$$
where S is the transverse size of the beam.

1.2 Moving mirror

The second mirror M_2 is now attached to a mechanical oscillator. $x_2 = L_0$ corresponds to the equilibrium position in the absence of light. Its position relative to L_0 is denoted x: $x_2 = L_0 + x$. Its mass is M, and it undergoes a restoring force $-M\Omega_M^2 x$, a friction force $-M\Gamma_M \dot{x}$ and the radiation pressure from the intra-cavity field.

1. Determine the steady state position x_s of the mirror.

2. We assume that the mirror displacement is small (as compared to what?), such that the cavity remains close to the resonance condition and that we can neglect the variations of Γ with x. Find a third power relation between the intra-cavity intensity I_{cav} and the incoming intensity I_{in}. Figure 2 gives the typical variations of $I_{\text{cav}} = f(I_{\text{in}})$. Show that the system presents a bistable behavior.

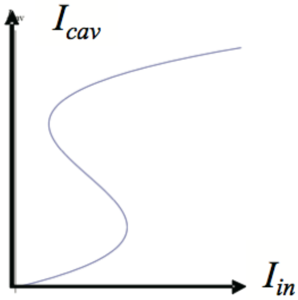

Figure 2: Typical behavior of $I_{\text{cav}} = f(I_{\text{in}})$.

2 Quantum field in a cavity with fixed mirrors

We now go to a model where the cavity field is quantized. The incoming laser is close to the cavity resonance at frequency ω_0, and the electric field associated to a single photon in the cavity is $\mathcal{E}_c = \sqrt{\hbar\omega_0/2\varepsilon_0 S L_0}$. The annihilation operator in the cavity mode is $\hat{a}$. The input laser field is in a coherent state, which complex amplitude is α_{in} in units of $\mathcal{E}_c$: $E_{\text{in}} = \alpha_{\text{in}}\mathcal{E}_c$.

2.1 Coupling between the cavity photons and the incoming field

1. If there is no loss in the cavity, explain why the system can be described by the following Hamiltonian:

$$\hat{H}_r = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar g \alpha_{\text{in}} \left(e^{i\omega t}\hat{a} + e^{-i\omega t}\hat{a}^\dagger \right), \quad \text{where } g \text{ is chosen such that} \quad g\alpha_{\text{in}} \in \mathbb{R}.$$

2. The cavity field state is described by a state vector

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} e^{-in\omega t}\,\tilde{c}_n(t)\,|n\rangle\,.$$

Give the equation of motion for the $\tilde{c}_n$ coefficients. Show that the state vector in the rotating frame $\left|\tilde{\psi}(t)\right\rangle = \sum_{n=0}^{\infty} \tilde{c}_n(t)\,|n\rangle$ obeys a Schrödinger equation for the effective Hamiltonian

$$\hat{H}_r' = -\hbar\Delta \hat{a}^\dagger \hat{a} + \hbar g \alpha_{\text{in}} \left(\hat{a} + \hat{a}^\dagger \right). \tag{3}$$

2.2 Cavity losses

We now take into account the imperfect reflection of the first mirror through a non zero γ.

1. Is the description in terms of state vector still appropriate?

2. We introduce the density matrix $\hat{\rho}$ of the intra-cavity field. What is the expression of the average value of an observable $\hat{A}$?

3. We recall the density matrix master equation in the Lindblad form[1]:

$$\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar}\left[\hat{H}, \hat{\rho}\right] + \Gamma\left(\hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{a}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{a}^\dagger\hat{a} \right). \tag{4}$$

[1]See Eq.(2.103) and (2.112) of the lecture notes.

Show that $\langle a \rangle$ obeys the differential equation

$$i\frac{d\langle a\rangle}{dt} = g\alpha_{\text{in}} - (\Delta + i\Gamma/2)\langle a\rangle\,. \tag{5}$$

4. Give the steady state α_{cav} of $\langle a\rangle$. By comparison with the classical solution, write the expression of the coupling constant g as a function of L_0 and γ.

5. Show that the density matrix $\hat{\rho}_s$ satisfying

$$\hat{a}\hat{\rho}_s = \alpha_{\text{cav}}\hat{\rho}_s \quad ; \quad \hat{\rho}_s\hat{a}^\dagger = \alpha^*_{\text{cav}}\hat{\rho}_s$$

is a possible steady state for the density matrix.

6. Show that the density matrix associated with a pure coherent state $|\alpha_{\text{cav}}\rangle$ is a solution of the previous equation. Conclusion?

3 Cavity with moving mirrors: radiation cooling of the mechanical mode

M_2 is now allowed to move around its equilibrium position. The cavity length at rest and no light is L_0, such that the instantaneous cavity length is $L_0 + x$. The mirror motion is quantized with the operators $\hat{x}$ and $\hat{p}$, with a position spread in the ground state $x_M = \sqrt{\hbar/2M\Omega_M}$.

3.1 Transition probability in the perturbative theory

1. Show that the coupled system {mirror + cavity field} is described by the Hamiltonian

$$\hat{H}_{\text{tot}} = \hat{H}'_r + \frac{\hat{p}^2}{2M} + \frac{1}{2}M\Omega_M^2\hat{x}^2 - \hbar f\hat{x}\hat{N},$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$ and f is a constant. Give the expression of f as a function of ω_0 and L_0. We assume that $\Gamma_M \ll \Gamma$ and neglect the mechanical damping in the following.

In order to describe the effect of the field fluctuations on the mirror, we use a Langevin model with the following Hamiltonian for the mirror:

$$\hat{H}_L = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\Omega_M^2\hat{x}^2 - \hbar f\hat{x}N(t).$$

$N(t)$ is now a classical stochastic function, with the same statistical properties as the quantum intra-cavity field of the last section.

2. Show that, with an appropriate change in the coordinate origin $x \to x'$, this Hamiltonian can be replaced, apart from a constant scalar, by

$$\hat{H}_L = \hbar\Omega_M\hat{b}^\dagger\hat{b} - \hbar f\hat{x}'\delta N(t), \tag{6}$$

where $\delta N(t) = N(t) - \overline{N}$ is the photon difference from the average steady state value $\overline{N}$ of the photon number calculated in the previous section. $\hat{b}$ and $\hat{b}^\dagger$ are the annihilation and creation operators of vibration quanta in the new coordinate system: $\hat{x}' = x_M(\hat{b} + \hat{b}^\dagger)$.

3. We assume that the mirror is initially in the Fock state $|m_0\rangle$ at $t = 0$ and we want to describe its subsequent evolution under the effect of $\hat{H}_I = -\hbar f\hat{x}'\delta N(t)$. The state vector is written under the form

$$|\psi(t)\rangle_M = \sum_m \lambda_m(t)\, e^{-im\Omega_M t}\,|m\rangle\,.$$

The coupling f is small enough to allow the use of the time-dependent perturbation theory with $\hat{H}_I$. Give the expression of λ_m as a time integral involving the matrix elements $\langle m|\hat{x}'|m_0\rangle$.

4. Show that, at first order of the perturbation theory, only the states $|m\rangle$ with $m = m_0 \pm 1$ are coupled to $|m_0\rangle$. What are the corresponding matrix elements?

5. Show that the average probability to find the mirror in state $|m_0 \pm 1\rangle$ after a time t is

$$P_\pm(t) = P_{m_0 \to m_0 \pm 1}(t) = \frac{2m_0 + 1 \pm 1}{2} f^2 x_M^2 \overline{\left| \int_0^t dt'\, e^{\pm i\Omega_M t'} \delta N(t') \right|^2}. \tag{7}$$

3.2 Fluctuation spectrum

The power spectrum density of a fluctuating signal $s(t)$ is defined as

$$S_s(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^T dt\, dt'\, \overline{\delta s^*(t)\delta s(t')}\, e^{i\omega(t-t')} = \int_{-\infty}^{\infty} d\tau\, \overline{\delta s^*(t)\delta s(t+\tau)}\, e^{-i\omega\tau}, \tag{8}$$

where $\delta s(t) = s(t) - \overline{s(t)}$.

1. Show that for sufficiently long times, the transition probability reads

$$P_\pm(t) \underset{t\to\infty}{\sim} t \frac{2m_0 + 1 \pm 1}{2} f^2 x_M^2\, S_N(\mp\Omega_M).$$

It follows that we have

$$\frac{dP_\pm}{dt} \underset{t\to\infty}{=} \frac{2m_0 + 1 \pm 1}{2} f^2 x_M^2\, S_N(\mp\Omega_M).$$

2. The power spectrum S_N of N can be shown to be

$$S_N(\Omega) = \frac{\overline{N}\,\Gamma}{(\Delta + \Omega)^2 + \Gamma^2/4},$$

see Fig. 3, and Eq.(3) of Ref. [3] or Eq.(8) of Ref. [4]. Show that for a negative detuning, the light fluctuations induce a reduction in the mirror vibration quanta.

3. What is the average value $\langle m\rangle$ of the mirror excitation number at long times, when thermal equilibrium is reached for the mirror? To which temperature[2] does is correspond? Find the optimum cooling detuning in the two limits $\Gamma \ll \Omega_M$ and $\Gamma \gg \Omega_M$. Comment.

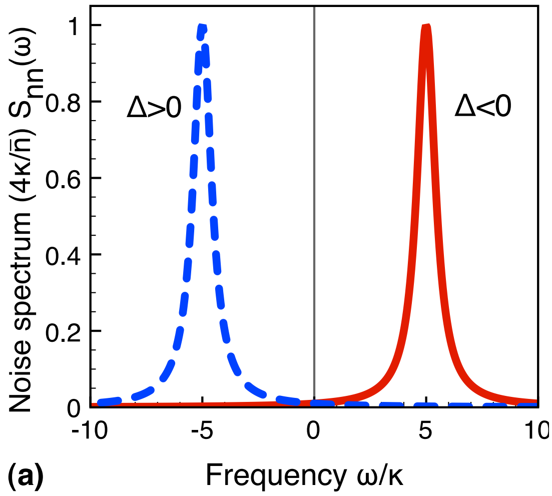

Figure 3: Noise spectrum of the photon number in the cavity, for two choices of the detuning $\Delta = \pm 5\Gamma$. Here, κ corresponds to our Γ. Figure from [3].

References

[1] Pierre Meystre. A short walk through quantum optomechanics. *Annalen der Physik*, 525(3):215–233, 2013.

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