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Quantum opto-mechanical devices — Answers

1 Classical field in a Fabry-Perot cavity

1.1 Fixed mirrors

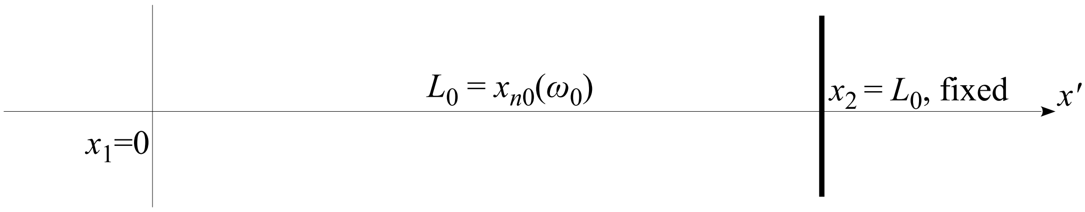

Figure 1: Fixed M_2 mirror. The origin of x' axis is at the position of mirror M_1. ω_0 is defined as $x_{n_0}(\omega_0) = L_0$, *i.e.* the resonant frequency for a cavity length L_0.

1. On resonance, the cavity length is a multiple of $\lambda/2$: $L_0 = n\lambda_n/2$. Using the relation $\omega_n = kc = 2\pi c/\lambda_n$, we deduce that the eigenfrequencies ω_n of the intra-cavity field read
$$\omega_n = n\frac{\pi c}{L_0}.$$
They are spaced by the free spectral range $\Delta\omega = \pi c/L_0$.

2. At first order in γ:
$$t_1 = (1 - r^2)^{1/2} = \left[1 - (1-\gamma)^2\right]^{1/2} = (2\gamma - \gamma^2)^{1/2} \simeq \sqrt{2\gamma}.$$

3. E^+_{cav} is the sum of all fields propagating to the right inside the cavity. The initial transmission through the first mirror M_1 gives a factor t. Each successive propagation to the right mirror M_2 and back, with 100% reflection, gives a phase shift $2kL_0$, and each successive reflection at M_1 gives a factor r. We have thus:
$$E^+_{\text{cav}} = tE_{\text{in}} + tre^{i2kL_0}E_{\text{in}} + t\left(re^{i2kL_0}\right)^2 E_{\text{in}} + \cdots = \frac{t}{1 - re^{i2kL_0}}E_{\text{in}}.$$
The laser frequency is close to a resonance at frequency $\omega_0 = n_0\pi c/L_0$. This means that the detuning is small as compared to the free spectral range: $\Delta \ll \Delta\omega$. Let us write explicitly the wave vector k:
$$2kL_0 = 2\frac{\omega}{c}L_0 = 2\frac{\omega_0}{c}L_0 + 2\frac{\Delta}{c}L_0 = 2n_0\pi + 2\frac{L_0}{c}\Delta.$$
The exponential term thus simplifies in $e^{i2L_0\Delta/c}$. We remark that the exponent also reads $i2\pi\Delta/\Delta\omega$ and is small. We can expand it and write
$$1 - re^{i2kL_0} = 1 - re^{i2\frac{L_0}{c}\Delta} \simeq 1 - r - ir2\frac{L_0}{c}\Delta \simeq \gamma - i2\frac{L_0}{c}\Delta.$$

We neglect the difference between r and 1 in the second term as $\Delta/\Delta\omega$ is already small. The relation between the fields thus reads:

$$E^+_{\rm cav} \simeq \frac{t}{\gamma - i2\frac{L_0}{c}\Delta} E_{\rm in} = \frac{itc/(2L_0)}{\Delta + i\frac{\gamma c}{2L_0}} E_{\rm in} = \frac{it/\tau}{\Delta + i\frac{\Gamma}{2}} E_{\rm in} \tag{1}$$

where we have defined $\tau = 2L_0/c = 2\pi/\Delta\omega$, the total round-trip time, and $\Gamma = \gamma c/L_0 = t^2/\tau$, the inverse photon lifetime in the cavity. Taking the modulus square, we can now write

$$\frac{I^+_{\rm cav}}{I_{\rm in}} = \frac{|E^+_{\rm cav}|^2}{|E_{\rm in}|^2} = \frac{t^2/\tau^2}{\Delta^2 + \frac{\Gamma^2}{4}} = \frac{4t^2/(\tau^2\Gamma^2)}{1 + 4\Delta^2/\Gamma^2} = \frac{4/t^2}{1 + 4\Delta^2/\Gamma^2}.$$

The field propagating along $-x$ results in a standing wave in the cavity, with an average intensity $I_{\rm cav} = 2I^+_{\rm cav}$. By identification, we find the relation

$$\frac{I_{\rm cav}}{I_{\rm in}} = \frac{A}{1 + 4\Delta^2/\Gamma^2}, \tag{2}$$

with $A = 8/t^2 = 4/\gamma$. A is related to the cavity finesse $\mathcal{F} = \Delta\omega/\Gamma = \pi/\gamma$ through $A = 4\mathcal{F}/\pi$.

4. Each photon reflected onto the mirror M_2 has its momentum reversed, from $+\hbar k$ to $-\hbar k$. The momentum change is thus $-2\hbar k$. Conversely, the mirror undergoes a momentum kick $+2\hbar k$ at each reflection. This phenomenon happens at a rate given by the number of photons impinging on the mirror per second, which is also the ratio between the power inside the cavity associated with the light propagating to the right $P^+_{\rm cav} = SI^+_{\rm cav} = SI_{\rm cav}/2$ and the energy per photon $\hbar\omega$. We can finally write:

$$F = 2\hbar k \frac{P^+_{\rm cav}}{\hbar\omega} = \frac{k}{\omega}\frac{2SI_{\rm cav}}{2} = \frac{SI_{\rm cav}}{c}. \tag{3}$$

1.2 Moving mirror

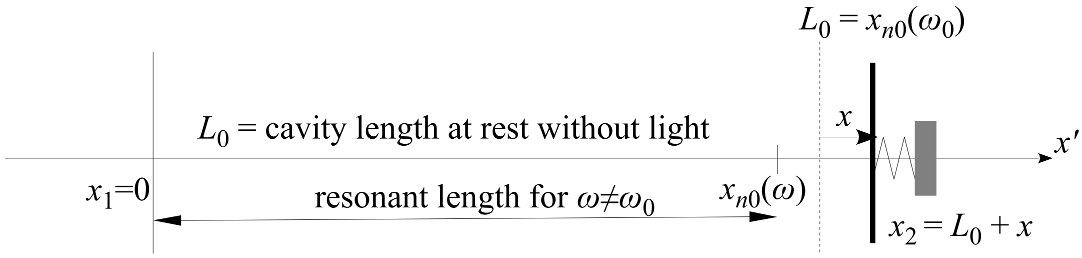

Figure 2: Moving M_2 mirror. L_0 is by definition the cavity length for a mirror at its equilibrium position, in the absence of light. ω_0 is defined as earlier, such that $x_{n_0}(\omega_0) = L_0$. At an driving frequency ω, the resonant length is $x_{n_0}(\omega)$. The position of mirror M_2 is x_2 with respect to M_1, and x with respect to L_0.

The laser frequency being given at ω, the mirror position where the cavity is resonant is: $x_2 = x_{n_0}(\omega) = n_0\lambda/2 = n_0\pi c/\omega = L_0\omega_0/\omega$. We will still use $\omega_0 = n_0\pi c/L_0$, but the resonance frequency $\omega_{\rm res}(x)$ for a given x now differs from ω_0 and depends on x. We have:

$$\omega_{\rm res}(x) = \frac{n_0\pi c}{x_2} = \frac{n_0\pi c}{L_0 + x} = \frac{\omega_0}{1 + x/L_0} \simeq \omega_0 - \omega_0\frac{x}{L_0}.$$

It is equal to ω_0 for $x = 0$. The detuning with respect to this new resonant frequency is

$$\Delta'(x) = \omega - \omega_{\rm res}(x) \simeq \Delta + \omega_0\frac{x}{L_0}$$

to lowest order in x/L_0.

1. The radiation pressure depends on x: modifying the cavity length changes the resonance frequency, and thus $\Delta'(x)$ and F. The dynamical equation for x is

$$M\ddot{x} = -M\Gamma_M \dot{x} - M\Omega_M^2 x + F(x).$$

In the steady state where the derivatives vanish, we have $x = x_s$ where

$$x_s(I_{\text{cav}}) = \frac{F}{M\Omega_M^2} = \frac{SI_{\text{cav}}}{M\Omega_M^2 c}. \tag{4}$$

2. The mirror displacement is small as compared to the wavelength. At first order, Γ is also modified in principle and reads $\Gamma' \simeq \Gamma(1 - x/L_0)$. We can write

$$\frac{2\Delta'(x)}{\Gamma'} \simeq \frac{2\Delta}{\Gamma}\left(1 + \frac{\omega_0}{\Delta}\frac{x}{L_0} + \frac{x}{L_0}\right) \simeq \frac{2\Delta}{\Gamma} + \frac{2\omega_0 x}{\Gamma L_0} = \frac{2\Delta}{\Gamma} + \frac{2k_0}{\gamma}x = \frac{2\Delta}{\Gamma} + K_0 x \tag{5}$$

where we used the expression of Γ and introduced $K_0 = 2k_0/\gamma$. We have eventually neglected the correction to Γ, which is of order x/L_0, smaller by a large factor ω_0/Δ as compared to the correction coming from the detuning. We can thus give the expression of the radiation pressure force $F(x)$ as a function of x:

$$F(x) = \frac{ASI_{\text{in}}}{c}\frac{1}{1 + \left(\frac{2\Delta}{\Gamma} + K_0 x\right)^2} = \frac{F_{\max}}{P_2(K_0 x)} \tag{6}$$

where $F_{\max} = ASI_{\text{in}}/c$ and $P_2(u) = \left(\frac{2\Delta}{\Gamma} + u\right)^2$ is a polynomial of order two.

Remark

We can even write more exactly the expression taking into account the modification of Γ: the new damping rate is related to the new cavity length $x_2 = L_0 + x$:

$$\Gamma'(x) = \frac{\gamma c}{L_0 + x} = \frac{\Gamma}{1 + x/L_0} \Leftrightarrow \Gamma'(x)^{-1} = \Gamma^{-1}\left(1 + \frac{x}{L_0}\right).$$

On the other hand, the exact value of $\Delta'(x)$ is

$$\Delta'(x) = \omega - \omega_{\text{res}}(x) = \omega - \frac{\omega_0}{1 + x/L_0}.$$

We get for the exact expression of the ratio $2\Delta'/\Gamma'$:

$$\frac{2\Delta'(x)}{\Gamma'(x)} = 2\Gamma^{-1}\left(1 + \frac{x}{L_0}\right)\left(\omega - \frac{\omega_0}{1 + x/L_0}\right) = 2\Gamma^{-1}\left(\omega - \omega_0 + \frac{\omega}{L_0}x\right) = \frac{2\Delta}{\Gamma} + Kx$$

with $K = 2\omega/(\Gamma L_0) = 2k/\gamma$. The exact version of Eq. (5) has a k instead of k_0 in the definition of K_0, we have thus just neglected a very small term $K_0 x\Delta/\omega_0$, smaller by a factor Δ/ω_0.

Using Eq. (4), we can then write Δ' as a function of I_{cav} at equilibrium:

$$\frac{2\Delta'(I_{\text{cav}})}{\Gamma} = \frac{2\Delta}{\Gamma} + Kx_s(I_{\text{cav}}) = \frac{2\Delta}{\Gamma} + \frac{KSI_{\text{cav}}}{M\Omega_M^2 c} = \frac{2\Delta}{\Gamma} + \frac{I_{\text{cav}}}{I_0} \tag{7}$$

where we have introduced the characteristic intensity

$$I_0 = \frac{M\Omega_M^2 c}{KS}.$$

$I_{\rm cav}$ is thus the solution of an equation of degree three $I_{\rm in}/I_0 = \tilde{f}(I_{\rm cav}/I_0)$ where $\tilde{f}$ is a polynomial of degree three:

$$\frac{I_{\rm cav}}{I_0} = \frac{A}{1+4\Delta'^2/\Gamma^2}\frac{I_{\rm in}}{I_0} = \frac{A}{1+\left(\dfrac{2\Delta}{\Gamma}+\dfrac{I_{\rm cav}}{I_0}\right)^2}\frac{I_{\rm in}}{I_0} = \frac{A}{P_2(I_{\rm cav}/I_0)}\frac{I_{\rm in}}{I_0}.$$

$$I_{\rm in} = I_0\tilde{f}(I_{\rm cav}/I_0) \quad \text{with} \quad \tilde{f}(u) = uP_2(u)/A.$$

There can be several solutions for the cavity intensity $I_{\rm cav}$ for a given value of $I_{\rm in}$. This leads to a bistable behavior, with an unstable region. If we write $I_{\rm in} = \tilde{f}(I_{\rm cav})$, the unstable region is determined by solving $\tilde{f}'(I_{\rm cav}) = 0$. If there is no solution or a single degenerate solution, which is the case if $\Delta > -\sqrt{3}\Gamma/2$, see below, the system is stable. If there are two solutions $I^{\pm}_{\rm cav}$, the system is unstable between $I^{+}_{\rm in}$ and $I^{-}_{\rm in}$, where $I^{\pm}_{\rm in} = \tilde{f}(I^{\pm}_{\rm cav})$.

Remark

Condition of stability: the radiation pressure is a function of x, and thus derives from a potential, just like the restoring force. Let us introduce the total potential $V(x)$ corresponding to the sum of both forces. Its derivative is linked to the forces

$$V'(x) = M\Omega_M^2 x - F(x) = M\Omega_M^2 x - \frac{F_{\max}}{P_2(Kx)} = M\Omega_M^2\left[x - \frac{A}{KP_2(Kx)}\frac{I_{\rm in}}{I_0}\right] = M\Omega_M^2 x\left[1 - \frac{1}{\tilde{f}(Kx)}\frac{I_{\rm in}}{I_0}\right].$$

It cancels for $x = x_s$, which is solution of $\tilde{f}(Ks_s) = I_{\rm in}/I_0$. This equilibrium position will be stable if the second derivative of the potential is positive at this point:

$$V''(x) = M\Omega_M^2\left[1 - \frac{1}{\tilde{f}(Kx)}\frac{I_{\rm in}}{I_0}\right] + M\Omega_M^2 Kx\frac{\tilde{f}'(Kx)}{\tilde{f}(Kx)^2}\frac{I_{\rm in}}{I_0}.$$

The first term cancels at the equilibrium condition, and as $x_s > 0$, the condition $V''(x_s) > 0$ reads

$$\tilde{f}'(Kx_s) > 0.$$

Remark Study of the unstable region. The steady state is obtained when $\tilde{f}(u) = I_{\rm in}/I_0$. Instabilities appear if there are several solutions for u. From (4), we know that the steady state is such that $u > 0$: as anticipated, the light pushes the mirror to the right. We then have to look for situations where there is a flex point $u_0 > 0$ where $\tilde{f}''(u_0) = 0$, and such that $\tilde{f}'(u_0) < 0$, which corresponds to the situation where 3 solutions exist in a range $[u_1, u_2]$, the edges being defined by $\tilde{f}'(u_1) = \tilde{f}'(u_2) = 0$. After simple arithmetics, we find that such a domain exists only if

$$\Delta < -\frac{\sqrt{3}}{2}\Gamma \tag{8}$$

and $u_0 = -4\Delta/(3\Gamma)$, see Fig. 3. The edges are given by

$$u_1 = \frac{2|\Delta|}{3\Gamma}\left(2 - \sqrt{1 - \frac{3\Gamma^2}{4\Delta^2}}\right) \qquad u_2 = \frac{2|\Delta|}{3\Gamma}\left(2 + \sqrt{1 - \frac{3\Gamma^2}{4\Delta^2}}\right) < \frac{2|\Delta|}{\Gamma}.$$

The u_1 position is where the system becomes unstable when the intensity is increased, whereas the u_2 position corresponds to the system being unstable when coming from above. As the resonant cavity length corresponds to $u_{\rm res} = -2\Delta/\Gamma = 2|\Delta|/\Gamma$, see e.g. (7), the jump by decreasing intensities occurs at the left of the resonant position, for $\Delta' < 0$.

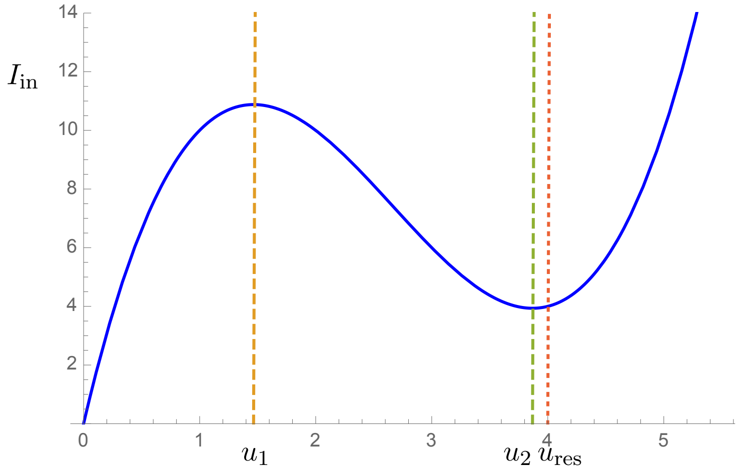

Figure 3: Representation of $\tilde{f}(u)$, the incoming intensity in units of I_0 as a function of the mirror position in units of K^{-1}, in the case $\Delta = -2\Gamma$. The points u_1 and u_2, with the corresponding intensities, are the edges of the unstable region. $u_{\rm res}$ is the length for which the cavity is resonant.

If, in the case where $\Delta < -\frac{\sqrt{3}}{2}\Gamma$, we increase the incoming intensity starting from $I_{\rm in} = 0$, we will start on the lower branch where the mirror shift is small. As $\Delta < 0$, increasing the cavity length brings the cavity closer to resonance with the laser. The radiation pressure thus increases more than linearly with $I_{\rm in}$. When the reduced position reaches u_1, the system becomes unstable and the mirror shifts abruptly to a new position, larger than u_2, and which may also be larger than $u_{\rm res}$: in this case the mirror has shifted on the other side where the detuning is positive. The more we increase the intensity, the more the cavity gets detuned, which ensures the system stability. Now, if we lower the intensity, the mirror shifts to the left. Below u_2, the system becomes unstable: the radiation pressure decreases too fast because the cavity gets detuned and is not able to compensate for the restoring force of the spring. It jumps to a position at the left of u_1, where both the restoring force and the radiation pressure are small, in a quasi linear regime.

2 Quantum field in a cavity with fixed mirrors

We now go to a model where the cavity field is quantized. The incoming laser is close to the cavity resonance at frequency ω_0, and the electric field associated to a single photon in the cavity is $\mathcal{E}_c = \sqrt{\hbar\omega_0/2\varepsilon_0 S L_0}$. The annihilation operator in the cavity mode is $\hat{a}$. The input laser field is in a coherent state, which complex amplitude is $\alpha_{\rm in}$ in units of $\mathcal{E}_c$: $E_{\rm in} = \alpha_{\rm in}\mathcal{E}_c$.

2.1 Coupling between the cavity photons and the incoming field

1. If there is no loss in the cavity, the system can be described by a Hamiltonian:

$$\hat{H}_r = \hbar\omega_0 \hat{a}^\dagger\hat{a} + \hbar g \alpha_{\rm in} e^{-i\omega t}\hat{a}^\dagger + h.c.$$

The first term is the energy of the harmonic oscillator describing the quantized cavity field, $\hat{a}^\dagger\hat{a}$ being the *total* photon number operator (corresponding to $I_{\rm cav}$ in the classical picture), while the second term describes a process where part of the incoming field $\alpha_{\rm in}e^{-i\omega t}$ is coupled through g into the cavity and creates a cavity photon. The hermitian conjugate describes the reverse process. If we don't make assumptions on the real character of $g\alpha_{\rm in}$, the field hamiltonian reads

$$\hat{H}_r = \hbar\omega_0 \hat{a}^\dagger\hat{a} + \hbar\left(g\alpha_{\rm in}e^{-i\omega t}\hat{a}^\dagger + g^*\alpha_{\rm in}^* e^{i\omega t}\hat{a}\right).$$

2. The cavity field state is described by a state vector

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} e^{-in\omega t}\,\tilde{c}_n(t)\,|n\rangle\,.$$

We write the Schrödinger equation for $|\psi(t)\rangle$:

$$i\hbar\partial_t\,|\psi(t)\rangle = \hat{H}_r\,|\psi(t)\rangle\,. \tag{9}$$

The left hand side is

$$i\hbar\partial_t\,|\psi(t)\rangle = \sum_{n=0}^{\infty} e^{-in\omega t}\left[\,i\hbar\partial_t\tilde{c}_n(t) + \,n\hbar\omega\tilde{c}_n(t)\right]|n\rangle\,.$$

We see that $\hat{H}_r\,|n\rangle = n\hbar\omega_0\,|n\rangle + \hbar\left(g\alpha_{\rm in}e^{-i\omega t}\sqrt{n+1}\,|n+1\rangle + g^*\alpha_{\rm in}^*e^{i\omega t}\sqrt{n}\,|n-1\rangle\right)$. The right hand side of Eq.(9) is thus:

$$\begin{aligned}
\hat{H}_r\,|\psi(t)\rangle &= \sum_{n=0}^{\infty} e^{-in\omega t}\,n\hbar\omega_0\tilde{c}_n(t)\,|n\rangle \\
&+\hbar g\alpha_{\rm in}\sum_{n=0}^{\infty} e^{-i(n+1)\omega t}\,\tilde{c}_n(t)\sqrt{n+1}\,|n+1\rangle + \hbar g^*\alpha_{\rm in}^*\sum_{n=1}^{\infty} e^{-i(n-1)\omega t}\,\tilde{c}_n(t)\sqrt{n}\,|n-1\rangle\,, \\
\hat{H}_r\,|\psi(t)\rangle &= \sum_{n=0}^{\infty} e^{-in\omega t}\,\left[n\hbar\omega_0\tilde{c}_n(t) + \hbar g\alpha_{\rm in}\sqrt{n}\tilde{c}_{n-1}(t) + \hbar g^*\alpha_{\rm in}^*\sqrt{n+1}\tilde{c}_{n+1}(t)\right]|n\rangle\,.
\end{aligned}$$

We can therefore write for each n:

$$i\hbar\partial_t\tilde{c}_n(t) = -n\hbar\Delta\tilde{c}_n(t) + \hbar g\alpha_{\rm in}\sqrt{n}\tilde{c}_{n-1}(t) + \hbar g^*\alpha_{\rm in}^*\sqrt{n+1}\tilde{c}_{n+1}(t).$$

It follows that the state vector in the rotating frame $\left|\tilde{\psi}(t)\right\rangle = \sum_{n=0}^{\infty}\tilde{c}_n(t)\,|n\rangle$ obeys a Schrödinger equation for the effective Hamiltonian

$$\hat{H}_r' = -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar g\alpha_{\rm in}\hat{a}^\dagger + \hbar g^*\alpha_{\rm in}^*\hat{a}. \tag{10}$$

2.2 Cavity losses

1. If dissipation through the coupling mirror is taken into account, the Hamiltonian description is not appropriate any more and we should switch to a description in terms of density matrices.

2. The average value of an observable $\hat{A}$ is $\langle A\rangle = {\rm Tr}(\hat{\rho}\hat{A})$.

3. We have by definition

$$i\frac{d\,\langle a\rangle}{dt} = i\frac{d}{dt}\left\{{\rm Tr}(\hat{\rho}\hat{a})\right\} = {\rm Tr}\left(i\frac{d\hat{\rho}}{dt}\hat{a}\right).$$

Then:

$$i\frac{d\,\langle a\rangle}{dt} = {\rm Tr}\left(\frac{1}{\hbar}\left[\hat{H}_r',\hat{\rho}\right]\hat{a}\right) + i\Gamma{\rm Tr}\left(\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \frac{1}{2}\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \frac{1}{2}\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}\right).$$

Using the property of the trace, which is invariant through cyclic permutation, the second part can be recast under the form:

$$i\Gamma{\rm Tr}\left(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \frac{1}{2}\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \frac{1}{2}\hat{\rho}\hat{a}^\dagger\hat{a}^2\right) = i\frac{\Gamma}{2}{\rm Tr}\left(\hat{\rho}\left[\hat{a}^\dagger,\hat{a}\right]\hat{a}\right) = -i\frac{\Gamma}{2}\langle a\rangle\,,$$

where we used $\left[\hat{a}, \hat{a}^\dagger\right] = 1$. Using again the property of the trace, the first term is

$$\frac{1}{\hbar}\mathrm{Tr}\left(\hat{H}'_r\hat{\rho}\hat{a} - \hat{\rho}\hat{H}'_r\hat{a}\right) = \frac{1}{\hbar}\mathrm{Tr}\left(\hat{\rho}\hat{a}\hat{H}'_r - \hat{\rho}\hat{H}'_r\hat{a}\right) = \left\langle \frac{1}{\hbar}\left[\hat{a}, \hat{H}'_r\right]\right\rangle.$$

Using $[\hat{a}, \hat{a}] = 0$ and $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, the commutator reads

$$\frac{1}{\hbar}\left[\hat{a}, \hat{H}'_r\right] = -\Delta\left[\hat{a}, \hat{a}^\dagger\hat{a}\right] + g\alpha_{\mathrm{in}} = -\Delta\left[\hat{a}, \hat{a}^\dagger\right]\hat{a} + g\alpha_{\mathrm{in}} = -\Delta\hat{a} + g\alpha_{\mathrm{in}}.$$

After taking the average, we finally obtain

$$i\frac{d\langle a\rangle}{dt} = g\alpha_{\mathrm{in}} - (\Delta + i\Gamma/2)\langle a\rangle. \quad (11)$$

4. The steady state α_{cav} of $\langle a\rangle$ is reached when $\frac{d\langle a\rangle}{dt}$ vanishes. We get

$$\alpha_{\mathrm{cav}} = \frac{g}{\Delta + i\Gamma/2}\alpha_{\mathrm{in}}.$$

   It measures the *total* field amplitude in the cavity. By analogy with the classical case, Eq. (1), and introducing a $\sqrt{2}$ factor to get the total field, we can identify the coupling constant g

$$g = i\sqrt{2}\frac{t}{\tau} = i\sqrt{\gamma}\frac{c}{L_0}.$$

5. Let use calculate the right hand side of the equation of motion of the density matrix $\hat{\rho}_s$. We have

$$\hat{a}\hat{\rho}_s\hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{a}\hat{\rho}_s - \frac{1}{2}\hat{\rho}_s\hat{a}^\dagger\hat{a} = \frac{1}{2}\left[\hat{a}\hat{\rho}_s, \hat{a}^\dagger\right] - \frac{1}{2}\left[\hat{\rho}_s\hat{a}^\dagger, \hat{a}\right] = \frac{\alpha_{\mathrm{cav}}}{2}\left[\hat{\rho}_s, \hat{a}^\dagger\right] - \frac{\alpha^*_{\mathrm{cav}}}{2}\left[\hat{\rho}_s, \hat{a}\right].$$

   The hamiltonian term gives

$$\frac{1}{\hbar}\left[\hat{H}'_r, \hat{\rho}_s\right] = -\frac{1}{\hbar}\left[\hat{\rho}_s, \hat{H}'_r\right] = \Delta\left[\hat{\rho}_s, \hat{a}^\dagger\hat{a}\right] - g\alpha_{\mathrm{in}}\left[\hat{\rho}_s, \hat{a}^\dagger\right] - g^*\alpha^*_{\mathrm{in}}\left[\hat{\rho}_s, \hat{a}\right].$$

   We now remark that $g\alpha_{\mathrm{in}} = \alpha_{\mathrm{cav}}\Delta + i\alpha_{\mathrm{cav}}\Gamma/2$ and $g^*\alpha^*_{\mathrm{in}} = \alpha^*_{\mathrm{cav}}\Delta - i\alpha^*_{\mathrm{cav}}\Gamma/2$. The terms in Γ will thus cancel and we can write

$$\begin{aligned} i\frac{d\hat{\rho}_s}{dt} &= \Delta\left[\hat{\rho}_s, \hat{a}^\dagger\hat{a}\right] - \Delta\alpha_{\mathrm{cav}}\left[\hat{\rho}_s, \hat{a}^\dagger\right] - \Delta\alpha^*_{\mathrm{cav}}\left[\hat{\rho}_s, \hat{a}\right] \\ &= \Delta\left(\left[\hat{\rho}_s, \hat{a}^\dagger\hat{a}\right] - \left[\hat{a}\hat{\rho}_s, \hat{a}^\dagger\right] - \left[\hat{\rho}_s\hat{a}^\dagger, \hat{a}\right]\right) \\ &= \Delta\left(\hat{\rho}_s\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}_s - \hat{a}\hat{\rho}_s\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\hat{\rho}_s - \hat{\rho}_s\hat{a}^\dagger\hat{a} + \hat{a}\hat{\rho}_s\hat{a}^\dagger\right) \\ &= 0. \end{aligned}$$

   $\hat{\rho}_s$ is therefore a possible steady state for the density matrix.

6. The density matrix associated with a pure coherent state $|\alpha_{\mathrm{cav}}\rangle$ is $\hat{\rho}_s = |\alpha_{\mathrm{cav}}\rangle\langle\alpha_{\mathrm{cav}}|$. It is clear that this density matrix verifies the two relations, as $\hat{a}\,|\alpha_{\mathrm{cav}}\rangle = \alpha_{\mathrm{cav}}\,|\alpha_{\mathrm{cav}}\rangle$ and $\langle\alpha_{\mathrm{cav}}|\,\hat{a}^\dagger = \alpha^*_{\mathrm{cav}}\,\langle\alpha_{\mathrm{cav}}|$. The state towards which the cavity relaxes is a coherent state associated to the amplitude α_{cav}.

3 Cavity with moving mirrors: radiation cooling of the mechanical mode

3.1 Transition probability in the perturbative theory

1. The coupled system {mirror + cavity field} is described by the Hamiltonian

$$\hat{H}_{\rm tot} = \hat{H}'_r + \frac{\hat{p}^2}{2M} + \frac{1}{2}M\Omega_M^2\hat{x}^2 - \hbar f \hat{x}\hat{N},$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$ and f is a constant. We have added to the field Hamiltonian the quantum mechanical oscillator and the coupling arising from the radiation pressure. $-\hbar f\hat{x}\hat{N}$ corresponds to the quantum version of the classical energy $E = -Fx = -SI_{\rm cav}x/c$. Let us now replace $I_{\rm cav}$ by its expression as a function of $\mathcal{E}_c$, using $\alpha_{\rm cav}\mathcal{E}_c = \sqrt{2}E^+_{\rm cav}$ ($\alpha_{\rm cav}$ describes the total field):

$$\frac{SI_{\rm cav}}{c} = 2\times 2\varepsilon_0 S|E^+_{\rm cav}|^2 = \frac{\hbar\omega_0}{L_0}\frac{2|E^+_{\rm cav}|^2}{\mathcal{E}_c^2} = \hbar\frac{\omega_0}{L_0}|\alpha_{\rm cav}|^2 = \hbar\frac{\omega_0}{L_0}\langle\hat{N}\rangle.$$

By identification, we can thus write

$$f = \frac{\omega_0}{L_0}.$$

2. We describe the field by a classical fluctuating variable $N(t) = \overline{N} + \delta N(t)$. The origin shift corresponds to chose the classical steady state x_s as the new origin. It corresponds to $x_s = \hbar f\overline{N}/(M\Omega_M^2)$. This doesn't modify the $\hat{p}$ operator. Writing $\hat{x} = x_s + \hat{x}'$, we obtain

$$\begin{aligned}
& \frac{1}{2}M\Omega_M^2(x_s+\hat{x}')^2 - \hbar f\hat{x}'N - \hbar f x_s N \\
= \;& \frac{1}{2}M\Omega_M^2\hat{x}'^2 + M\Omega_M^2 x_s\hat{x}' + \frac{1}{2}M\Omega_M^2 x_s^2 - \hbar f\hat{x}'N - \hbar f x_s N \\
= \;& \frac{1}{2}M\Omega_M^2\hat{x}'^2 - \hbar f\hat{x}'\delta N + \hbar f x_s\left(\frac{\overline{N}}{2} - N\right).
\end{aligned}$$

Apart from scalar terms, we end up for the oscillator Hamiltonian with

$$\hat{H}_L = \hbar\Omega_M\hat{b}^\dagger\hat{b} - \hbar f\hat{x}'\delta N(t) = \hbar\Omega_M\hat{b}^\dagger\hat{b} - \hbar f x_M\delta N(t)(\hat{b}+\hat{b}^\dagger). \tag{12}$$

Remark We could instead write $\hat{a} = \alpha_{\rm cav} + \delta\hat{a}$ and $\hat{a}^\dagger = \alpha^*_{\rm cav} + \delta\hat{a}^\dagger$, in the spirit of a Bogoliubov approach were we linearize the field around a classical field, and keep only the second order terms in the fluctuations. This is what is done in Ref. [1], Eq.(2). Then $x_s = \hbar f|\alpha_{\rm cav}|^2/(M\Omega_M^2)$. We also have

$$\begin{aligned}
\hat{x}\hat{N} &\simeq (x_s+\hat{x}')\left(\alpha^*_{\rm cav}+\delta\hat{a}^\dagger\right)(\alpha_{\rm cav}+\delta\hat{a}) \\
&\simeq |\alpha_{\rm cav}|^2 x_s + |\alpha_{\rm cav}|^2\hat{x}' + \alpha_{\rm cav}x_s\delta\hat{a}^\dagger + \alpha^*_{\rm cav}x_s\delta\hat{a} + \alpha_{\rm cav}\hat{x}'\delta\hat{a}^\dagger + \alpha^*_{\rm cav}\hat{x}'\delta\hat{a} + x_s\delta\hat{a}^\dagger\delta\hat{a}
\end{aligned}$$

where we have neglected the third order term. The first term is constant, the second one cancels the shift of the harmonic oscillator center. The Hamiltonian, apart from a constant term, reads

$$\hat{H} = \hat{H}'_r + \hbar\Omega_M\hat{b}^\dagger\hat{b} - \hbar f\left[(\alpha_{\rm cav}\delta\hat{a}^\dagger + \alpha^*_{\rm cav}\delta\hat{a})x_s + x_M(\alpha_{\rm cav}\delta\hat{a}^\dagger + \alpha^*_{\rm cav}\delta\hat{a})(\hat{b}+\hat{b}^\dagger) + x_s\delta\hat{a}^\dagger\delta\hat{a}\right].$$

The terms is x_s can be reabsorbed in the definition of the detuning $\Delta' = \Delta + f x_s$, as in Eq. (5). This gives finally:

$$\hat{H} = -\hbar\Delta'\delta\hat{a}^\dagger\delta\hat{a} + \hbar\Omega_M\hat{b}^\dagger\hat{b} - \hbar f x_M(\alpha_{\rm cav}\delta\hat{a}^\dagger + \alpha^*_{\rm cav}\delta\hat{a})(\hat{b}+\hat{b}^\dagger).$$

3. Let us write the action of $i\hbar\partial_t$, $\hat{H}_0$ and $\hat{H}_I$ on $|\psi(t)\rangle_M$:

$$i\hbar\partial_t |\psi(t)\rangle_M = \sum_m \left(i\hbar\dot{\lambda}_m + m\hbar\Omega_M\lambda_m\right) e^{-im\Omega_M t} |m\rangle ,$$
$$\hat{H}_0 |\psi(t)\rangle_M = \sum_m m\hbar\Omega_M\lambda_m \, e^{-im\Omega_M t} |m\rangle ,$$
$$\hat{H}_I |\psi(t)\rangle_M = -\hbar f\delta N(t) \sum_m \lambda_m \, e^{-im\Omega_M t}\hat{x}' |m\rangle .$$

The Schrödinger equation leads to

$$\sum_m i\hbar\dot{\lambda}_m \, e^{-im\Omega_M t} |m\rangle = -\hbar f\delta N(t) \sum_{m'} \lambda_{m'} \, e^{-im'\Omega_M t}\hat{x}' |m'\rangle ,$$

or

$$\dot{\lambda}_m = if\delta N(t) \sum_{m'} \lambda_{m'} \, e^{-i(m'-m)\Omega_M t} \langle m| \hat{x}' |m'\rangle .$$

We limit ourselves to the first order. We should take the zeroth order for $\lambda_{m'}$ in the sum, that is $\lambda^{(0)}_{m'} = \delta_{m_0,m'}$. Writing the equation in the integral form, we finally have

$$\lambda_m(t) = if \langle m| \hat{x}' |m_0\rangle \int_0^t e^{i(m-m_0)\Omega_M t'} \delta N(t')dt'.$$

4. $\hat{x}'$ only couples $|m_0\rangle$ with $|m_0 \pm 1\rangle$, with $\langle m_0 \pm 1| \hat{x}' |m_0\rangle = x_M\sqrt{m_0 + \frac{1}{2} \pm \frac{1}{2}}$.

5. The probability to find the mirror in state $|m_0 \pm 1\rangle$ after a time t is

$$P_\pm(t) = |\lambda_{m_0\pm 1}(t)|^2 = \frac{2m_0 + 1 \pm 1}{2} f^2 x_M^2 \left| \int_0^t dt' \, e^{\pm i\Omega_M t'} \delta N(t') \right|^2 . \tag{13}$$

3.2 Fluctuation spectrum

The power spectrum density of a fluctuating signal $s(t)$ is defined as

$$S_s(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_0^T\!\!\int_0^T dt \, dt' \, \overline{\delta s^*(t)\delta s(t')} \, e^{i\omega(t-t')} = \int_{-\infty}^{\infty} d\tau \, \overline{\delta s^*(t)\delta s(t+\tau)} \, e^{-i\omega\tau}, \tag{14}$$

where $\delta s(t) = s(t) - \overline{s(t)}$.

1. To obtain the probability at long times, we should take the statistical average and let t go to infinity. We can write:

$$\overline{\left| \int_0^t dt' \, e^{\pm i\Omega_M t'} \delta N(t') \right|^2} = \int_0^t dt' \int_0^t dt'' \, e^{\mp i\Omega_M (t'-t'')} \overline{\delta N^*(t')\delta N(t'')}.$$

At large t, this is equivalent by definition to

$$\sim tS_N(\mp\Omega_M).$$

It follows that we have

$$\frac{dP_\pm}{dt} \underset{t\to\infty}{=} \frac{2m_0 + 1 \pm 1}{2} f^2 x_M^2 \, S_N(\mp\Omega_M).$$

2. The power spectrum S_N of N can be shown to be

$$S_N(\Omega) = \frac{\overline{N}\,\Gamma}{(\Delta + \Omega)^2 + \Gamma^2/4}.$$

For a negative detuning, $S_N(-\Omega_M)$ is smaller than $S_N(+\Omega_M)$. Provided this is not compensated by the m_0 or $m_0 + 1$ pre-factor which goes the other way round, the rate to reach a lower state $\dot{P}_-$ is larger than the rate for reaching a higher vibrational state. As a consequence, the mirror vibration is cooled down.

3. This process will stop for a average $\langle m \rangle$ such that the energy is stationary:

$$\begin{aligned}
&\dot{E} = \left\langle \dot{P}_+ \right\rangle \hbar\Omega_M - \left\langle \dot{P}_- \right\rangle \hbar\Omega_M = 0 \\
&\Rightarrow (\langle m \rangle + 1) S_N(-\Omega_M) = \langle m \rangle\, S_N(\Omega_M) \\
&\Rightarrow \frac{\langle m \rangle + 1}{\langle m \rangle} = \exp\left[\frac{\hbar\Omega_M}{k_B T}\right] = \frac{S_N(\Omega_M)}{S_N(-\Omega_M)} = \frac{(\Delta - \Omega_M)^2 + \Gamma^2/4}{(\Delta + \Omega_M)^2 + \Gamma^2/4}.
\end{aligned}$$

We deduce the equilibrium temperature

$$k_B T = \frac{\hbar\Omega_M}{\ln\left(\frac{(\Delta - \Omega_M)^2 + \Gamma^2/4}{(\Delta + \Omega_M)^2 + \Gamma^2/4}\right)}. \tag{15}$$

The temperature is minimum if we chose

$$\Delta = -\sqrt{\Omega_M^2 + \frac{\Gamma^2}{4}}.$$

The average excitation number corresponding to this choice is

$$\langle m \rangle = \frac{\sqrt{1 + \frac{\Gamma^2}{4\Omega_M^2}} - 1}{2}.$$

In the limit $\Gamma \gg \Omega_M$ where the vibrational structure is not resolved, this value corresponds to what we also find for laser cooling on a transition of width Γ. The limit temperature is $k_B T = \hbar\Gamma/4$. In the other limit $\Gamma \ll \Omega_M$, the laser can induce Raman resolved sideband cooling, and the optimum choice is to repeat a cycle where a laser photon is absorbed and a photon of larger frequency is emitted into the cavity mode, which reduces the vibrational energy by $\hbar\Omega_M$ each time. The limit temperature is then

$$k_B T_{\rm lim} = \frac{\hbar\Omega_M}{2\ln\left(\frac{4\Omega_M}{\Gamma}\right)}.$$

It is much less than $\hbar\Omega_M$ and corresponds to a large population in the ground state:

$$n_0 = 1 - e^{-\frac{\hbar\Omega_M}{k_B T}} \simeq 1 - \frac{\Gamma^2}{16\Omega_M^2}.$$

References

[1] I. Wilson-Rae, N. Nooshi, W. Zwerger, and T. J. Kippenberg. Theory of ground state cooling of a mechanical oscillator using dynamical backaction. *Phys. Rev. Lett.*, 99:093901, Aug 2007.