Atoms dressed by an electromagnetic field – Answers

I. Fluorescence in the strong coupling regime

We consider a two-level atom, ground state $|g\rangle$ and excited state $|e\rangle$, interacting with a strong monochromatic electromagnetic field at frequency ω: $\Omega_1 \gg \Gamma$, where Ω_1 is the Rabi frequency characterizing the coupling between the atom and the field, $\hbar\omega_0$ the energy of the excited state $|e\rangle$ and Γ its inverse life time. We will study the fluorescence spectrum of the atom in this strong coupling limit. The excitation field is a strong laser field, close to the atomic resonance at ω_0. We are interested only in the atomic internal states and we won't consider the external state in this problem.

1 Laser field quantization

In a first step, we completely neglect the finite life time of the excited state ($\Gamma = 0$). Although the excitation field is classical in the sense that the mean photon number $\langle N\rangle$ interacting with the atoms is very large, and so is its dispersion ΔN, its relative fluctuations $\Delta N/\langle N\rangle$ are negligible and we will gain a deeper insight in the coupling by using a quantized description of the field. This will make much clearer the interpretation of the fluorescence spectrum. The laser field is thus in a coherent state $|\alpha\rangle$, where we chose $\alpha \in \mathbb{R}$.

We start from the expression of the classical field of the plane wave of polarization $\boldsymbol{\epsilon}$ at fixed position:

$$\mathbf{E}(t) = \mathcal{E}_{\rm cl}\boldsymbol{\epsilon}\, e^{-i\omega t} + c.c.$$

1. Give the relation between α and $\langle N\rangle$ and ΔN.

Answer: The average number of photons in a coherent state is

$$\langle N\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2$$

where we used the fact that $|\alpha\rangle$ is an eigenstate of $\hat{a}$. As for ΔN, we have

$$\begin{aligned}\Delta N^2 &= \langle N^2\rangle - \langle N\rangle^2 = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle - |\alpha|^4 \\ &= \langle\alpha|(\hat{a}^\dagger)^2\hat{a}^2|\alpha\rangle + \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle - |\alpha|^4 = |\alpha|^4 + |\alpha|^2 - |\alpha|^4 = |\alpha|^2 = \langle N\rangle \qquad (1)\end{aligned}$$

where we have used $\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$. The dispersion is thus $\Delta N = |\alpha| = \sqrt{\langle N\rangle}$. This justifies why $\Delta N/N = 1/\sqrt{\langle N\rangle} \ll 1$.

2. Show that the quantum field can be described as follows:

$$\hat{\mathbf{E}} = \mathcal{E}_0\left(\boldsymbol{\epsilon}\hat{a} + \boldsymbol{\epsilon}^*\hat{a}^\dagger\right) \qquad (2)$$

and give the relation between $\mathcal{E}_{\rm cl}$ and $\mathcal{E}_0$.

Answer: From Eq. (3.124) of the lecture notes, and taking a different phase origin to remove the i factor from this equation, we have

$$\hat{\mathbf{E}} = \sum_{\ell} \mathcal{E}_\ell \left(\boldsymbol{\epsilon}_\ell \hat{a}_\ell e^{i\mathbf{k}\cdot\mathbf{r}} + \boldsymbol{\epsilon}_\ell^* \hat{a}_\ell^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right)$$

where $\mathcal{E}_\ell = \sqrt{\hbar\omega_\ell/(2\epsilon_0\mathcal{V})}$. The summation is for each laser mode ℓ, and $\mathcal{V}$ is the quantization volume. Here, we only have one laser mode, and we finally get

$$\hat{\mathbf{E}} = \mathcal{E}_0 \left(\boldsymbol{\epsilon} \hat{a} + \boldsymbol{\epsilon}^* \hat{a}^\dagger \right)$$

where the possible phase factor in $\mathcal{E}_0$ is included in $\boldsymbol{\epsilon}$. For a classical state we have $\mathcal{E}_{\text{cl}} = \alpha\mathcal{E}_0$.

3. We recall the dipole operator $\hat{\mathbf{d}} = \mathbf{d}\hat{\sigma}_+ + \mathbf{d}^*\hat{\sigma}_-$ where $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$. Give the expression of the atom-field Hamiltonian, introducing the single photon Rabi frequency Ω_0. Relate it to the average Rabi frequency of the coherent state Ω_1.

Answer: The atom-field hamiltonian at the dipolar approximation is

$$\hat{H}_{\text{AF}} = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}} = -\mathcal{E}_0 \left(\mathbf{d} \cdot \boldsymbol{\epsilon} \hat{a} \hat{\sigma}_+ + h.c. \right) - \mathcal{E}_0 \left(\mathbf{d} \cdot \boldsymbol{\epsilon}^* \hat{a}^\dagger \hat{\sigma}_+ + h.c. \right).$$

The second term is non resonant and is neglected in the rotating wave approximation (RWA). The single photon Rabi frequency is

$$\hbar\Omega_0 = -2\mathcal{E}_0 \mathbf{d} \cdot \boldsymbol{\epsilon}.$$

We choose the time origin in such a way that $\Omega_0 \in \mathbb{R}^+$.

In RWA, the Hamiltonian writes

$$\hat{H}_{\text{RWA}} = \frac{\hbar\Omega_0}{2} \hat{a}\hat{\sigma}_+ + h.c.$$

This Hamiltonian couples the state $|g, \alpha\rangle = |g\rangle \otimes |\alpha\rangle$ to $|e, \alpha\rangle$ with the amplitude

$$\langle e, \alpha | \hat{H}_{\text{RWA}} | g, \alpha \rangle = -\mathcal{E}_0 \mathbf{d} \cdot \boldsymbol{\epsilon} \alpha = \frac{\hbar\Omega_0}{2}\alpha = \frac{\hbar\Omega_1}{2}.$$

We can thus identify $\Omega_1 = \alpha\Omega_0 = \sqrt{\langle N \rangle}\Omega_0$ as $\alpha \in \mathbb{R}$.

4. Write the full Hamiltonian including the atomic internal energy, the quantum monochromatic field and the interaction term.

Answer: We will denote the non-RWA coupling as $\Omega_0' = -2\mathcal{E}_0 \mathbf{d} \cdot \boldsymbol{\epsilon}^*$. With a proper choice of energy reference, the total Hamiltonian reads

$$\hat{H} = \frac{\hbar\omega_0}{2} \left(|e\rangle\langle e| - |g\rangle\langle g| \right) + \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \frac{\hbar\Omega_0}{2} \left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_- \right) + \frac{\hbar\Omega_0'}{2} \hat{a}^\dagger \hat{\sigma}_+ + \frac{\hbar\Omega_0'^*}{2} \hat{a}\hat{\sigma}_-.$$

2 Uncoupled states

1. In the absence of coupling (for $\Omega_0 = 0$), write the eigenstate of the {atom + photons} system and their energy.

Answer: Without coupling, the eigenstates are the states $|e, N\rangle$ and $|g, N\rangle$, where N is the photon number in the laser mode. Their energies are

$$E_{e,N-1} = \frac{\hbar\omega_0}{2} + \left(N - 1 + \frac{1}{2}\right)\hbar\omega = -\frac{\hbar\delta}{2} + N\hbar\omega \qquad \text{for } N \geq 1,$$
$$E_{g,N} = -\frac{\hbar\omega_0}{2} + \left(N + \frac{1}{2}\right)\hbar\omega = \frac{\hbar\delta}{2} + N\hbar\omega \qquad \text{for } N \geq 0,$$

where we have defined the detuning $\delta = \omega - \omega_0$.

2. Show that, for a near resonant frequency, they organized into multiplicities $\mathcal{E}_N$ of dimension 2. Give a sketch of the levels.

Answer: Near resonance frequency means that $|\delta| \ll \omega, \omega_0$. The two energies of states $|e, N-1\rangle$ and $|g, N\rangle$ differ by only δ, whereas the energy difference between these states and the closest other states are of order $\hbar\omega$. The multiplicity $\mathcal{E}_N$ is thus defined as

$$\mathcal{E}_N = \{|e, N-1\rangle, |g, N\rangle\}.$$

N.B.: The $\mathcal{E}_0$ manifold contains only one state, the ground state $|g, 0\rangle$. The levels inside the multiplicity are separated by $\hbar\delta$. The center of the multiplicity is separated by $\hbar\omega$ from the next multiplicities $\mathcal{E}_{N-1}$ and $\mathcal{E}_{N+1}$, see Fig. 1.

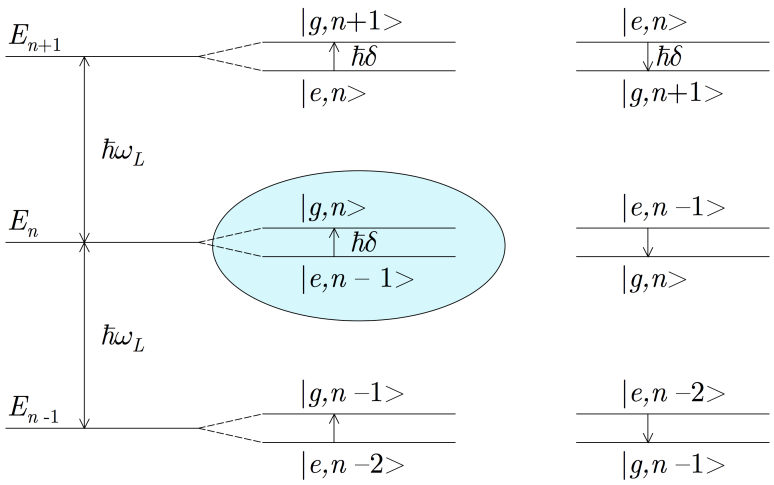

Figure 1: Multiplicities $\mathcal{E}_N$ for the uncoupled atom-photon system. Left: $\delta > 0$. Right: $\delta < 0$.

3 Dressed states

Now, the coupling is non zero.

1. Identify the states of the uncoupled basis which are now coupled by the field. Do they belong to the same multiplicity?

Answer: The interaction Hamiltonian $\hat{H}_{\rm AF}$ couples the states inside a given multiplicity $\mathcal{E}_N$:

$$\langle e, N-1| \hat{H}_{\rm AF} |g, N\rangle = \langle g, N| \hat{H}_{\rm AF} |e, N-1\rangle = \frac{\hbar\Omega_0}{2}\sqrt{N}.$$

It also couples states of multiplicities N and N' separated by $N' - N = \pm 2$:

$$\langle e, N+1| \hat{H}_{\rm AF} |g, N\rangle = \frac{\hbar\Omega_0'}{2}\sqrt{N+1}$$

couples the $\mathcal{E}_N$ manifold to the $\mathcal{E}_{N+2}$ manifold, whereas

$$\langle g, N-2| \hat{H}_{\rm AF} |e, N-1\rangle = \frac{\hbar{\Omega_0'}^*}{2}\sqrt{N-1}$$

couples the $\mathcal{E}_N$ manifold to the $\mathcal{E}_{N-2}$ manifold.

2. Give their coupling amplitude. We recall that the photon number is very large as compared to 1.

Answer: As $\langle N\rangle \gg 1$ and $\Delta N/\langle N\rangle \ll 1$, we can write $\sqrt{N} \simeq \sqrt{N+1} \simeq \sqrt{N-1} \simeq \sqrt{\langle N\rangle}$. The coupling are thus $\hbar\Omega_1/2$ within each $\mathcal{E}_N$ and $\Omega_1'/2$ or ${\Omega_1'}^*/2$ between multiplicities with $\Delta N = 2$.

3. Give an estimation of the energy shift induced by the two types of coupling, to lowest order in the perturbation theory. Under which assumption can we restrict ourselves to a given manifold?

Answer: Let us look at the perturbed state $|g, N\rangle^{(1)}$ in the first order perturbation theory:

$$\begin{aligned}|g, N\rangle^{(1)} &= |g, N\rangle + \sum_{\psi} \frac{\langle\psi| \hat{H}_{\rm AF} |g, N\rangle}{E_{g,N} - E_\psi} |\psi\rangle \\ &= |g, N\rangle + \frac{\Omega_1}{2\delta} |e, N-1\rangle - \frac{\Omega_1'}{2(\omega+\omega_0)} |e, N+1\rangle .\end{aligned}$$

In the limit where $|\delta| \ll \omega_0$ and $\Omega_1' \ll \omega, \omega_0$, it is clear that the last term (coupling outside a given manifold) is negligible. This gives us the relevant assumptions to restrict our description to a manifold $\mathcal{E}_N$, all manifolds with $N \simeq \langle N\rangle$ behaving in the same way.

4. Diagonalize the Hamiltonian inside a given manifold $\mathcal{E}_N$ and give the expression of the dressed states $|\pm, N\rangle$ and their energies $E_{\pm,N}$. How does the state evolve in a frequency sweep from a large and negative detuning to a large and positive detuning?

Answer: The Hamiltonian inside $\mathcal{E}_N$ writes (see question 1 and recall $\Omega_1 \in \mathbb{R}$)

$$\hat{H} = \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_1 \\ \Omega_1 & \delta \end{pmatrix} + N\hbar\omega.$$

The eigenenergies are deduced as follows:

$$\frac{\hbar}{2}\begin{vmatrix} -\delta-\lambda & \Omega_1 \\ \Omega_1 & \delta-\lambda \end{vmatrix} = 0 \Longleftrightarrow \lambda^2 - \delta^2 - \Omega_1^2 = 0 \Longleftrightarrow \lambda = \pm\sqrt{\delta^2+\Omega_1^2}.$$

The eigenenergies are

$$E_{\pm,N} = \pm\frac{\hbar}{2}\sqrt{\delta^2+\Omega_1^2} + N\hbar\omega.$$

Let's find the eigenstates $|\pm, N\rangle = a_\pm |e, N-1\rangle + b_\pm |g, N\rangle$:

$$\hat{H}\,|\pm, N\rangle = E_{\pm,N}\,|\pm, N\rangle \iff -\delta a_\pm + \Omega_1 b_\pm = \pm\sqrt{\delta^2+\Omega_1^2}\,a_\pm.$$

We define the angle θ as

$$\cos\theta = \frac{-\delta}{\sqrt{\delta^2+\Omega_1^2}} \qquad \sin\theta = \frac{\Omega_1}{\sqrt{\delta^2+\Omega_1^2}}. \tag{3}$$

$a_\pm$ and $b_\pm$ satisfy

$$(\cos\theta \mp 1)a_\pm = -\sin\theta\, b_\pm \iff (1 \mp \cos\theta)a_\pm = \pm\sin\theta\, b_\pm.$$

From this relation we deduce the eigenstates

$$|+, N\rangle = \cos\frac{\theta}{2}\,|e, N-1\rangle + \sin\frac{\theta}{2}\,|g, N\rangle\,, \tag{4}$$

$$|-, N\rangle = -\sin\frac{\theta}{2}\,|e, N-1\rangle + \cos\frac{\theta}{2}\,|g, N\rangle\,. \tag{5}$$

The upper level is connected to $|g, N\rangle$ for a large and positive detuning (when $\theta \simeq \pi$), and to $|e, N-1\rangle$ for a large and negative detuning (when $\theta \simeq 0$).

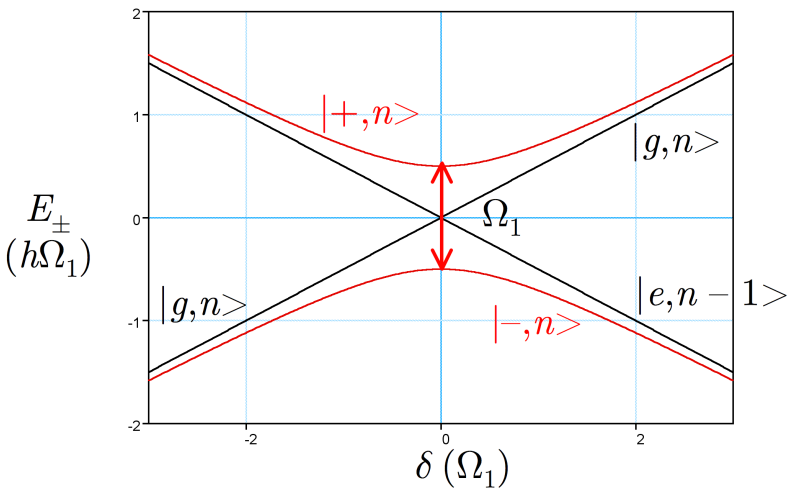

Figure 2: Energies of the uncoupled states and of the dressed states $|\pm, N\rangle$, in units of $\hbar\Omega_1$, as a function of the detuning δ in units of Ω_1. The offset energy $N\hbar\omega$ has been removed.

5. Calculate the energy in the limit of a large detuning. Comment.

Answer: As we mentioned earlier, when $|\delta| \gg \Omega_1$, $|g, N\rangle \simeq |\varepsilon, N\rangle$ where ε is the sign of δ. The energy shift of the ground state is then approximately

$$\Delta E_g \simeq E_{\varepsilon,N} - E_{g,N} = \varepsilon \frac{\hbar}{2}\sqrt{\delta^2 + \Omega_1^2} - \frac{\hbar\delta}{2} = \frac{\hbar\delta}{2}\sqrt{1 + \frac{\Omega_1^2}{\delta^2}} - \frac{\hbar\delta}{2} \simeq \frac{\hbar\Omega_1^2}{4\delta}.$$

The ground state energy is shifted by $\hbar\Omega_1^2/(4\delta)$. We recover the usual expression for the light shift in the large detuning limit, see the TD on light forces.

6. Why is the adiabatic sweep an interesting way to transfer the population from $|g\rangle$ to $|e\rangle$ in the presence of inhomogeneous broadening?

Answer: Flipping the internal state from $|g\rangle$ to $|e\rangle$ through Rabi oscillations requires to have a π pulse, with a well-defined duration τ such that $\sqrt{\Omega_1^2 + \delta^2}\tau = \pi$. If, however, there is a dispersion $\Delta\omega_0$ in the atomic frequencies ω_0, due for example to their velocity distribution or to some inhomogeneous environment, only part of the atoms will fulfill the π condition while the others will have a non zero probability to stay in state $|g\rangle$.

Consider now a sweep of the detuning from a negative value such that $|\delta(t = 0)| \gg \Omega_1, \Delta\omega_0$ to a final positive value such that $\delta(t = \tau) \gg \Omega_1, \Delta\omega_0$. When the coupling is switched on at large negative detuning, $|g\rangle$ is projected essentially onto $|-, N\rangle$, as the initial detuning has be chosen large also with respect to the frequency inhomogeneity. If the frequency sweep is adiabatic, the atoms will stay in the $|-, N\rangle$ state until the end of the pulse, where now this state is projected essentially onto $|e, N-1\rangle$ for the same reason. In this way, all the atoms are transferred from $|g\rangle$ to $|e\rangle$, even in the presence of frequency inhomogeneities.

7. Give the condition for an adiabatic following of the dressed state during the frequency sweep. We recall from the adiabatic theorem that the overlap of $\partial_t |1\rangle$, the variation of an adiabatic state $|1\rangle$, with any other adiabatic state $|2\rangle$ must remain very small as compared to the frequency splitting between the states:

$$|\langle 2| \partial_t |1\rangle| \ll \frac{|E_1 - E_2|}{\hbar}.$$

Answer: Consider as before that we want to follow the state $|-, N\rangle$. From (5), we can write:

$$\partial_t |-, N\rangle = -\frac{\dot\theta}{2}\left(\cos\frac{\theta}{2}|e, N-1\rangle + \sin\frac{\theta}{2}|g, N\rangle\right) = -\frac{\dot\theta}{2}|+, N\rangle.$$

As the two states coupled by the time evolution are split by $\sqrt{\delta^2 + \Omega_1^2}$, we finally get

$$|\dot\theta| \ll 2\sqrt{\delta^2 + \Omega_1^2} \Longleftrightarrow \left|\dot\delta\Omega_1 - \delta\dot\Omega_1\right| \ll 2\left(\delta^2 + \Omega_1^2\right)^{3/2}.$$

4 Finite life time

We now take into account the finite life time Γ^{-1} of the excited state.

1. To which state will $|+, N\rangle$ and $|-, N\rangle$ decay due to spontaneous emission?

Answer: The finite lifetime affects the state $|e\rangle$, which can decay to $|g\rangle$. The coupled stated are thus $|e, N-1\rangle$ and $|g, N-1\rangle$, the latter being in the $\mathcal{E}_{N-1}$ manifold. The decay will thus couple the $|\pm, N\rangle$ states of $\mathcal{E}_N$ to the $|\pm, N-1\rangle$ states of $\mathcal{E}_{N-1}$.

2. What are the expected transition frequencies? Why is this fluorescence called the Mollow triplet?

Answer: There are three different transition frequencies, hence the Mollow triplet denomination:

$$\begin{aligned}
&|+, N\rangle \to |+, N-1\rangle \text{ and } |-, N\rangle \to |-, N-1\rangle \text{ at } \omega;\\
&|+, N\rangle \to |-, N-1\rangle \text{ at } \omega_+ = \omega + \sqrt{\delta^2 + \Omega_1^2};\\
&|-, N\rangle \to |+, N-1\rangle \text{ at } \omega_- = \omega - \sqrt{\delta^2 + \Omega_1^2};
\end{aligned}$$

The transition at ω is an elastic scattering of the laser photons. The two other lines change the nature of the state from $|-\rangle$ to $|+\rangle$ or conversely. They have to alternate in time.

3. What are the corresponding rates?

Answer: The rates are proportional to the coupling square (Fermi golden rule), and is at most Γ in the limit $|\delta| \gg \Omega_1$ where the dressed states connect to the atomic states.

Let us estimate the reduced dipole element $d_{s's}$ between the initial state $|s, N\rangle$ and the final state $|s', N-1\rangle$, where $s, s' = \pm$.

$$d_{s's} = \langle s', N-1 | \, (\hat{\sigma}_+ + \hat{\sigma}_-) \otimes \mathbb{1} \, |s, N\rangle$$

The dipole can then only couple states $|e, N-1\rangle$ (from $|s, N\rangle$) and $|g, N-1\rangle$ (from $|s', N-1\rangle$), which have the same laser photon number. As a consequence, only the second term $\hat{\sigma}_-$ contributes and we have:

$$d_{s's} = \langle s', N-1 | \, \hat{\sigma}_- \, |s, N\rangle \Longrightarrow \begin{cases} d_{\pm\pm} = \pm \cos\dfrac{\theta}{2} \sin\dfrac{\theta}{2} \\ d_{+-} = -\sin^2(\theta/2) \\ d_{-+} = \cos^2(\theta/2). \end{cases}$$

The transition rate from $|s, N\rangle$ to $|s', N-1\rangle$ is proportional to the square reduced dipole $d_{s's}^2$. For $\theta = 0$, the line width must be equal to Γ for the transition $+ \to -$, corresponding in fact to the $e \to g$ transition. The line width are thus

$$\Gamma_{++} = \Gamma_{--} = \Gamma \cos^2 \tfrac{\theta}{2} \sin^2 \tfrac{\theta}{2}$$

$$\Gamma_{+-} = \Gamma \sin^4 \tfrac{\theta}{2}$$

$$\Gamma_{-+} = \Gamma \cos^4 \tfrac{\theta}{2}.$$

N.B. The total decay rate from state $|+, N\rangle$ is $\Gamma_+ = \Gamma_{++} + \Gamma_{-+} = \cos^2 \frac{\theta}{2} \, \Gamma$, proportional to the weight of state $|+, N\rangle$ on the excited state $|e, N-1\rangle$.

4. Calculate, from rate equations, the steady state populations in the states $|+\rangle$ and $|-\rangle$.

Answer: The rate equation on the population π_+ of all states $|+, N\rangle$ reads

$$\dot{\pi}_+ = \Gamma_{++}\pi_+ + \Gamma_{+-}\pi_- - \Gamma_{++}\pi_+ - \Gamma_{-+}\pi_+ = \Gamma_{+-}\pi_- - \Gamma_{-+}\pi_+ = \Gamma_{+-} - (\Gamma_{+-} + \Gamma_{-+})\,\pi_+,$$

where we used $\pi_+ + \pi_- = 1$. The steady state solution is

$$\pi_+ = \frac{\Gamma_{+-}}{\Gamma_{+-} + \Gamma_{-+}} = \frac{\sin^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2} + \cos^4\frac{\theta}{2}} = \frac{\sin^4\frac{\theta}{2}}{1 - 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} = \frac{(1-\cos\theta)^2}{(1-\cos\theta)^2 + (1+\cos\theta)^2}.$$

5. Give the result in the limit where $|\delta| \gg \Omega_1$ and comment.

Answer: In the case where $\delta < 0$ and $|\delta| \gg \Omega_1$, we have $\theta \ll 1$ and $\pi_+ \simeq \theta^4/16$ and $\pi_- \simeq 1$. As $|-, N\rangle$ is connected to $|g, N\rangle$ in this limit, we find that the atoms are essentially in the ground state. In the same way, if $\delta > 0$ and $\delta \gg \Omega_1$, $\pi_- \simeq \theta^4/16$ and $\pi_+ \simeq 1$. Again, this is $|+, N\rangle$ which is connected to $|g, N\rangle$ now, and the atoms are in the ground state. In this situation, the scattering rate is very low, and the result is a pure light shift of the ground state.

6. Same question in the limit where $|\delta| \ll \Omega_1$.

Answer: This corresponds to the limit $\theta \simeq \pi/2$. Here, we have $\pi_+ \simeq \pi_- \simeq 1/2$. The atoms are in a superposition of $|e\rangle$ and $|g\rangle$, and jump from manifold to manifold, with emission of photons alternatively at frequencies ω_+ and ω_-.

Remark
We can calculate the total scattering rate $\Gamma_{\rm sc}$ as follows:

$$\Gamma_{\rm sc} = \Gamma_+\pi_+ + \Gamma_-\pi_- = \Gamma\frac{\cos^2\frac{\theta}{2}\sin^4\frac{\theta}{2} + \sin^2\frac{\theta}{2}\cos^4\frac{\theta}{2}}{1 - 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} = \frac{\Gamma}{2}\frac{\sin^2\theta}{2 - \sin^2\theta} = \frac{\Gamma}{2}\frac{1-\cos^2\theta}{1+\cos^2\theta}.$$

Using the expression of $\cos\theta$ as a function of δ and Ω_1, we finally recover

$$\Gamma_{\rm sc} = \frac{\Gamma}{2}\frac{\Omega_1^2/2}{\Omega_1^2/2 + \delta^2} = \frac{\Gamma}{2}\frac{s_0}{1+s_0}$$

where s_0 is the saturation parameter, equal to $s_0 = \Omega_1^2/(2\delta^2)$ in the limit $\Omega_1 \gg \Gamma$.

II. Radio-frequency adiabatic potentials

In this problem, we will consider the dressing of atoms with a radio-frequency (rf) field. This technique is used to tailor trapping potentials for atoms in a dressed state, using a position dependent magnetic field [1–4]. The example we can keep in mind is the rubidium 87 atom in its $F = 1$ state, in a static field of about $B_0 = 100\ \mu$T, in the presence of a rf field of a few hundred kHz up to 10 MHz, produced by an alternative current circulating in a loop. We recall the Landé factor $g_F = 1/2$ in the ground state of rubidium such that $g_F\mu_B/\hbar = 2\pi \times 7$ GHz·T^{-1}. We will assume for simplicity that $g_F > 0$.

1 Interaction between a spin and a classical magnetic field

1.1 Static magnetic field

We consider an atom with a total spin $\hat{\mathbf{S}}$. Recall the eigenstates $|m\rangle_z$ and the energies of the spin projection $\hat{S}_z = \hat{\mathbf{S}} \cdot \mathbf{e}_z$ in a static magnetic field $\mathbf{B}_0 = B_0 \, \mathbf{e}_z$. Recall the effects of the $\hat{S}_\pm$ operators on the states, where

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y. \tag{6}$$

Answer: The magnetic dipolar interaction Hamiltonian reads

$$\hat{H}_0 = -\gamma \hat{\mathbf{S}} \cdot \mathbf{B}_0 = \frac{g_F \mu_B}{\hbar} B_0 \hat{S}_z.$$

Its eigenstates are the common eigenstates $|m\rangle_z$ of $\hat{S}_z$ and $\hat{\mathbf{S}}^2$, with eigenvalues $\hbar m$ and $\hbar^2 S(S+1)$ respectively. This gives for the eigenenergies $E_m = m g_F \mu_B B_0 = m\hbar\omega_0$ where ω_0 is the Larmor frequency corresponding to B_0. The effect of $\hat{S}_\pm$ on these eigenstates is

$$\hat{S}_\pm |m\rangle_z = \hbar\sqrt{S(S+1) - m(m\pm 1)}\, |m\pm 1\rangle_z .$$

1.2 Spin in a rf classical field

In this part, we describe the rf field as a classical field. In a first step, we consider atoms with a spin S in a homogeneous magnetic field $\mathbf{B}_0$. The direction of the static field is chosen as quantization axis z: $\mathbf{B}_0 = B_0 \, \mathbf{e}_z$.

1. Explain why, at rf frequencies, spontaneous emission can be ignored.

Answer: The spontaneous scattering rate is proportional to the cube of the transition frequency. In the rf range, the decay rate is reduced by a factor $\sim (10^{15}/10^9)^3 = 10^{18}$, and can thus be ignored. We remark also that the ground state sublevels of rubidium are not coupled through dipolar electric coupling (same value of L) but only through dipolar magnetic coupling.

2. Give the corresponding Hamiltonian for the spin in the presence of a static field $B_0 \, \mathbf{e}_z$ and a homogeneous rf field $\mathbf{B}_1(t)$.

Answer: In the same way as for question 1.1, the total Hamiltonian reads

$$\hat{H} = \frac{g_F \mu_B}{\hbar} B_0 \hat{S}_z + \frac{g_F \mu_B}{\hbar} \mathbf{B}_1(t) \cdot \hat{\mathbf{S}} = \omega_0 \hat{S}_z + \frac{g_F \mu_B}{\hbar} \mathbf{B}_1(t) \cdot \hat{\mathbf{S}}.$$

The classical rf magnetic field can be written very generally as

$$\mathbf{B}_1 = B_x e^{-i\omega t} \, \mathbf{e}_x + B_y e^{i\phi} e^{-i\omega t} \, \mathbf{e}_y + B_z e^{i\phi_z} e^{-i\omega t} \, \mathbf{e}_z + c.c. \tag{7}$$

3. Explain why the z component of the rf field doesn't couple the $|m\rangle_z$ state to another spin state. We will discard this term[1] from now on.

[1] In fact, misalignment effects of the rf field, where there is a non zero component along the static field, do have a small effect, see [5].

Answer: The z component of the rf field projects onto $\hat{S}_z$, which is diagonal in the $\{|m\rangle_z\}$ basis. It doesn't couple the different $|m\rangle_z$ states (see however footnote).

4. We introduce the spherical basis $(\mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_z)$:

$$\mathbf{e}_+ = -\frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y) \qquad \mathbf{e}_- = \frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y). \tag{8}$$

Give the expression of $\hat{\mathbf{S}} \cdot \mathbf{e}_\pm$.

Answer:

$$\hat{\mathbf{S}} \cdot \mathbf{e}_+ = -\frac{1}{\sqrt{2}}\left(\hat{\mathbf{S}} \cdot \mathbf{e}_x + i\hat{\mathbf{S}} \cdot \mathbf{e}_y\right) = -\left(\hat{S}_x + i\hat{S}_y\right)/\sqrt{2} = -\frac{1}{\sqrt{2}}\hat{S}_+.$$

In the same way, $\hat{\mathbf{S}} \cdot \mathbf{e}_- = \frac{1}{\sqrt{2}}\hat{S}_-$.

5. Give the complex projections $B_\pm$ of the rf field amplitude onto this basis.

Answer: The projection onto $\mathbf{e}_+$ is obtained by taking the scalar product with $\mathbf{e}_+^* = -\mathbf{e}_-$ (the normalization rule is $\mathbf{e}_+^* \cdot \mathbf{e}_+ = 1$). We get

$$B_+ = \mathbf{e}_+^* \cdot \left(B_x\,\mathbf{e}_x + B_y e^{i\phi}\,\mathbf{e}_y\right) = \frac{1}{\sqrt{2}}\left(-B_x + iB_y e^{i\phi}\right),$$
$$B_- = \mathbf{e}_-^* \cdot \left(B_x\,\mathbf{e}_x + B_y e^{i\phi}\,\mathbf{e}_y\right) = \frac{1}{\sqrt{2}}\left(B_x + iB_y e^{i\phi}\right).$$

6. Write the spin Hamiltonian under the form

$$\hat{H} = \omega_0 \hat{S}_z + \left[\frac{\Omega_+}{2}\, e^{-i\omega t}\, \hat{S}_+ + \frac{\Omega_-}{2}\, e^{-i\omega t}\, \hat{S}_- + h.c.\right]. \tag{9}$$

Give the expression of $\Omega_\pm$.

Answer: Using $\mathbf{B}_1 = (B_+\mathbf{e}_+ + B_-\mathbf{e}_- + B_{1,z}\mathbf{e}_z)\, e^{-i\omega t} + c.c.$, the Larmor frequency ω_0 introduced earlier, and neglecting the effect of the z component of $\mathbf{B}_1$, we can write

$$\hat{H} = \omega_0 \hat{S}_z - \frac{g_F \mu_B}{\hbar}\frac{B_+}{\sqrt{2}} e^{-i\omega t}\hat{S}_+ + \frac{g_F \mu_B}{\hbar}\frac{B_-}{\sqrt{2}} e^{-i\omega t}\hat{S}_- + h.c.$$

which is equivalent to (9) provided that we set

$$\Omega_+ = -\sqrt{2}\frac{g_F \mu_B}{\hbar} B_+ = \frac{g_F \mu_B}{\hbar}\left(B_x - iB_y e^{i\phi}\right) \quad \text{and} \quad \Omega_- = \sqrt{2}\frac{g_F \mu_B}{\hbar} B_-.$$

We remark that, if the field is produced with two antennas of axes x and y with the same amplitude $B_x = B_y = B_{\rm rf}$, the coupling Ω_+ is maximum for a dephasing $\phi = \pi/2$ between the two antennas, and in this cas $\Omega_- = 0$. This corresponds to a σ^+ polarization of the rf field. On the other hand, if $\phi = -\pi/2$, Ω_- is maximum and Ω_+ vanishes.

7. Show that, in a frame rotating at frequency ω, and if we apply the rotating wave approximation, the spin Hamiltonian reads

$$\hat{H}_{\rm eff} = -\delta \hat{S}_z + \left[\frac{|\Omega_+|}{2}\, \hat{S}_+ + h.c.\right] = -\delta \hat{S}_z + |\Omega_+|\hat{S}_x. \tag{10}$$

$\delta = \omega - \omega_0$ is the detuning from resonance.

Answer: We introduce the state $\left|\tilde{\psi}(t)\right\rangle$ in the interaction picture

$$|\psi(t)\rangle = \hat{R}_z(\omega t)\left|\tilde{\psi}(t)\right\rangle = e^{-i\omega t\hat{S}_z/\hbar}\left|\tilde{\psi}(t)\right\rangle.$$

$\hat{R}_z(\omega t)$ is the rotation operator of angle ωt around the z axis. The time evolution of $|\psi\rangle$ writes

$$i\hbar\partial_t|\psi\rangle = \omega\hat{S}_z e^{-i\omega t\hat{S}_z/\hbar}\left|\tilde{\psi}\right\rangle + i\hbar e^{-i\omega t\hat{S}_z/\hbar}\partial_t\left|\tilde{\psi}\right\rangle = \hat{H}|\psi\rangle = \hat{H}e^{-i\omega t\hat{S}_z/\hbar}\left|\tilde{\psi}\right\rangle$$
$$\omega\hat{S}_z\left|\tilde{\psi}\right\rangle + i\hbar\partial_t\left|\tilde{\psi}\right\rangle = e^{i\omega t\hat{S}_z/\hbar}\hat{H}e^{-i\omega t\hat{S}_z/\hbar}\left|\tilde{\psi}\right\rangle = \hat{R}_z(\omega t)^\dagger\hat{H}\hat{R}_z(\omega t).$$

$\left|\tilde{\psi}\right\rangle$ thus obeys a Schrödinger equation with an effective Hamiltonian $H_{\text{eff}} = \hat{R}_z(\omega t)^\dagger\hat{H}\hat{R}_z(\omega t) - \omega\hat{S}_z$. Let us explicit this Hamiltonian. The rotation operator commutes with $\hat{S}_z$. Its effect on $\hat{S}_\pm$ is $\hat{R}_z(\omega t)^\dagger\hat{S}_\pm\hat{R}_z(\omega t) = e^{\pm i\omega t}\hat{S}_\pm$. We then have

$$H_{\text{eff}} = (\omega_0 - \omega)\hat{S}_z + \left[\frac{\Omega_+}{2}\,\hat{S}_+ + \frac{\Omega_-}{2}\,e^{-i2\omega t}\,\hat{S}_- + h.c.\right]. \tag{11}$$

Applying the RWA amounts to neglect the fast rotating term (in Ω_-) in the interaction picture. We are left with only the terms in Ω_+: only the σ^+ polarization component of the field contributes to the coupling. Moreover, if we include the phase of Ω_+ in the rotation around z, we can the Hamiltonian under the form (10).

8. Write the Hamiltonien under the form:

$$\hat{H}_{\text{eff}} = \sqrt{\delta^2 + |\Omega_+|^2}\left(\cos\theta\hat{S}_z + \sin\theta\hat{S}_x\right). \tag{12}$$

   Comment.

Answer: The diagonalization of this Hamiltonian is just what we've done in the first part. Using the expression of θ given at (3) with the correspondence $\Omega_1 \leftrightarrow |\Omega_+|$, we find the proposed expression.

9. Give the associated energies and eigenstates using spin rotation operators. What are the effect of both polarization components? Give the coupling amplitude in terms of B_x, B_y, and ϕ.

Answer: For a 1/2 spin, we've done it already. Can we be more general? The answer is positive. The effective Hamiltonian is just a spin Hamiltonian in a magnetic field with Larmor frequency $\Omega_L = \sqrt{\delta^2 + |\Omega_+|^2}$, pointing in a direction $\cos\theta\mathbf{e}_z + \sin\theta\mathbf{e}_x$, which is the image of $\mathbf{e}_z$ in a rotation of angle θ around the y axis. The eigenstates are thus the $|m\rangle_z$ states rotated by θ around y, $|\psi'\rangle = |m\rangle_\theta = \hat{R}_y(\theta)|m\rangle_z$, with eigenenergies $m\hbar\Omega_L$.

2 Adiabatic potentials

2.1 Adiabaticity

We now assume that the magnetic field $\mathbf{B}_0(\mathbf{r})$ is position dependent, in amplitude and in orientation. The rf field remains homogeneous.

1. In a first step, we treat the position $\mathbf{r}$ classically, as an external parameter in the Hamiltonian, and we assume that the atoms will follow adiabatically an eigenstate of the effective Hamiltonian. Give the expression of the adiabatic potential resulting from the interaction between the spin and the rf field.

Answer: The adiabatic potential in state $|m\rangle_\theta$ is the energy of the dressed state, dependent on position:

$$V_m(\mathbf{r}) = m\hbar\sqrt{\delta^2(\mathbf{r}) + |\Omega_+(\mathbf{r})|^2} = m\hbar\sqrt{(\omega - \omega_0(\mathbf{r}))^2 + |\Omega_+(\mathbf{r})|^2}$$

2. We first consider a static magnetic field with a fixed direction, and with a position-dependent amplitude. Show that the atoms with $m > 0$ are confined to an isomagnetic surface. What is the minimum energy spacing between adiabatic states?

Answer: Ω_+ is position independent because the quantization direction is fixed, which means that the eigenstates do not depend on position. The minimum $m\hbar|\Omega_+|$ is obtained on the points where $\delta(\mathbf{r}) = 0$ or $\omega_0(\mathbf{r}) = \omega$. This corresponds to the isomagnetic surface, where the Larmor frequency is equal to the rf frequency.

3. Give the expression of the oscillation frequency of this confinement in the harmonic approximation. We will denoted $b'(\mathbf{r}) = |\boldsymbol{\nabla} B_0|$ the local gradient of the magnetic field amplitude at position $\mathbf{r}$.

Answer: Consider a point $\mathbf{r}_0$ on the resonant surface. We have

$$B_0(\mathbf{r}) \simeq B_0(\mathbf{r}_0) + \boldsymbol{\nabla} B_0(\mathbf{r}_0) \cdot (\mathbf{r} - \mathbf{r}_0).$$

$$\omega_0(\mathbf{r}) \simeq \omega + \frac{g_F \mu_B b'(\mathbf{r}_0)}{\hbar} \frac{\boldsymbol{\nabla} B_0(\mathbf{r}_0)}{|\boldsymbol{\nabla} B_0|} \cdot (\mathbf{r} - \mathbf{r}_0).$$

Let use define $\alpha_0 = g_F \mu_B b'(\mathbf{r}_0)/\hbar$. The potential along the direction of the field gradient in the vicinity or $\mathbf{r}_0$ reads

$$V_m(\mathbf{r}) \simeq m\hbar\sqrt{\alpha_0^2 |\mathbf{r} - \mathbf{r}_0|^2 + |\Omega_+|^2} \simeq m\hbar|\Omega_+| + \frac{1}{2}\alpha_0^2 \frac{m\hbar}{|\Omega_+|} |\mathbf{r} - \mathbf{r}_0|^2.$$

This corresponds to a harmonic motion with an oscillation frequency

$$\omega_{\text{osc}} = \alpha_0 \sqrt{\frac{m\hbar}{M|\Omega_+|}}. \tag{13}$$

4. Numerics: We give typical figure of experiments with rf adiabatic potentials: $b' \simeq 5$ T·m^{-1}, $|\Omega_+|/(2\pi) = 30$ kHz, spin $F = 1$ for rubidium 87 where $h/M = 4.6 \times 10^{-9}$ Hz·m^2. Evaluate the oscillation frequency. Comment.

Answer: We have $\alpha_0 = g_F \mu_B/\hbar \times b' = 2\pi \times 7\,10^9 \times 5 = 2\pi \times 3.5\,10^{10}$ Hz·m^{-1}. Plugging this value into (13), we find $\omega_{\text{osc}} = 2\pi \times 2.2$ kHz. This is already a large oscillation frequency compared to typical magnetic traps.

5. We relax the assumption of a fixed magnetic field direction. If the direction of the static field varies in space slowly as compared to its amplitude, the previous conclusion remains valid. The change in the field orientation will determine the position of the trap minimum inside the isomagnetic surface.

   If the atomic position and velocity depend on time, the eigenstates will depend on time through $\theta(t)$. Explain why the adiabatic condition reads

$$|\dot{\theta}| \ll |\Omega_+|. \tag{14}$$

The adiabatic condition is similar to the one we've met already in the two-level case. The time derivative of $|m\rangle_\theta$ is proportional to $\hat{R}_y(\theta)\hat{S}_y\,|m\rangle_z$, coupled to $|m-1\rangle_\theta$ and $|m+1\rangle_\theta$, and the projection is of order $\dot{\theta}$, which must be small as compared to Ω_L, which is at least $|\Omega_+|$, hence the result. Detailed calculation:

$$\begin{aligned}
&\partial_t\,|m\rangle_\theta = \partial_t \hat{R}_y(\theta)\,|m\rangle_z = \partial_t e^{-i\theta\hat{S}_y/\hbar}\,|m\rangle_z = -\frac{i}{\hbar}\dot{\theta}\,e^{-i\theta\hat{S}_y/\hbar}\hat{S}_y\,|m\rangle_z \\
&= -\frac{\dot{\theta}}{2\hbar}\,\hat{R}_y(\theta)\left(\hat{S}_+ - \hat{S}_-\right)|m\rangle_z \\
&= -\frac{\dot{\theta}}{2}\,\hat{R}_y(\theta)\left(\sqrt{S(S+1)-m(m+1)}\,|m+1\rangle_z - \sqrt{S(S+1)-m(m-1)}\,|m-1\rangle_z\right) \\
&= -\frac{\dot{\theta}}{2}\left(\sqrt{S(S+1)-m(m+1)}\,|m+1\rangle_\theta - \sqrt{S(S+1)-m(m-1)}\,|m-1\rangle_\theta\right).
\end{aligned}$$

In particular, if we start in the upmost state $|m\rangle_\theta = |S\rangle_\theta$, we have

$$\partial_t\,|S\rangle_\theta = \frac{\dot{\theta}}{2}\sqrt{2S}\,|S-1\rangle_\theta = \dot{\theta}\sqrt{\frac{S}{2}}\,|S-1\rangle_\theta\,.$$

The adiabatic condition is thus

$$|\dot{\theta}|\sqrt{\frac{S}{2}} \ll \sqrt{\delta^2+|\Omega_+|^2}.$$

It is verified if $|\dot{\theta}| \ll |\Omega_+|$.

2.2 Example: the double-well trap on an atom chip

Using atom chips, where micro-wires are deposited on a substrates, it is possible to bring atoms very close to the wires and reach strong magnetic gradients. By an appropriate choice of wire configuration, we can create a static magnetic field which is the same as in a Ioffe-Pritchard trap:

$$\mathbf{B}_0(\mathbf{r}) = (B_{\min} + \frac{b''}{2}z^2)\,\mathbf{e}_z + b'(x\,\mathbf{e}_x - y\,\mathbf{e}_y) = B_z(z)\,\mathbf{e}_z + b'(x\,\mathbf{e}_x - y\,\mathbf{e}_y).$$

1. What are the isomagnetic surfaces in this configuration?

Answer: Isomagnetic surfaces are ellipsoids for magnetic fields close to $B_{\min}$.

$$B^2 = B_z^2(z) + b'^2(x^2+y^2) \simeq B_{\min}^2 + B_{\min}b''z^2 + b'^2(x^2+y^2).$$

For the isomagnetic surface corresponding to a field B_0, we have

$$B = B_0 \Leftrightarrow B_{\min}b''z^2 + b'^2(x^2+y^2) = B_0^2 - B_{\min}^2 \Leftrightarrow \frac{z^2}{R_z^2} + \frac{x^2+y^2}{r_\perp^2} = 1$$

where the radii are $R_z = \sqrt{B_0^2 - B_{\min}^2}/\sqrt{B_{\min}b''}$ and $r_\perp = \sqrt{B_0^2 - B_{\min}^2}/b'$. With typical values, we have $r_\perp \ll R_z$.

2. We add an rf field, linearly polarized along x: $\mathbf{B}_1(t) = B_1\,e^{-i\omega t}\,\mathbf{e}_x$. Explain qualitatively how the coupling strength $|\Omega_+|$ varies along the isomagnetic surface.

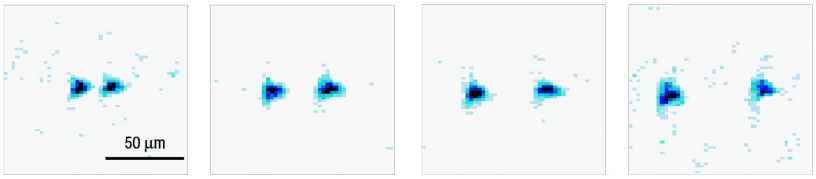

Figure 3: First experimental realization of the rf double well trap [6]. Two Bose-Einstein condensates are trapped in the double well, and imaged along the z direction.

Answer: Only the projection of $\mathbf{B}_1$ in a plane orthogonal to the local $\mathbf{B}_0$ contributes to the coupling, see question 3 of §1.2. $|\Omega_+|$ is thus minimum at $y = z = 0$ where the static field has the maximum component around x. It is maximum in the $x = 0$ plane where $\mathbf{B}_1$ is orthogonal to $\mathbf{B}_0(\mathbf{r})$.

More precisely, for a linearly polarized field (dephasing $\phi = 0$), the relevant coupling is $\hbar\Omega_+ = g_F\mu_B\sqrt{B_X^2 + B_Y^2}$ with a local (X, Y, Z) basis defined such that Z is the quantization axis aligned with the static magnetic field $\mathbf{B}_0$. In other word $\hbar\Omega_+ = g_F\mu_B B_\perp$ where $B_\perp$ is the projection of $\mathbf{B}_1$ in a plane orthogonal to the local $\mathbf{B}_0$. We can thus write

$$B_\perp = |\mathbf{B}_1 \times \mathbf{e}_Z| = \left|\mathbf{B}_1 \times \frac{\mathbf{B}_0}{|\mathbf{B}_0|}\right|.$$

With the Ioffe-Pritchard field and a rf polarization $B_1\mathbf{e}_x$ along x, it gives

$$B_\perp = \frac{B_1}{B_0(\mathbf{r})}\left|B_z(z)\mathbf{e}_x \times \mathbf{e}_z + \mathbf{0} - b'y\mathbf{e}_x \times \mathbf{e}_y\right| = \frac{B_1}{B_0(\mathbf{r})}\sqrt{B_z(z)^2 + b'^2y^2}.$$

On the isomagnetic surface defined by $g_F\mu_B B_0(\mathbf{r}) = g_F\mu_B B_{\rm res} = \hbar\omega$, is varies with x as

$$B_\perp = \frac{B_1}{B_{\rm res}}\sqrt{B_{\rm res}^2 - b'^2x^2} = B_1\sqrt{1 - \frac{b'^2x^2}{B_{\rm res}}}.$$

3. Deduce that this configuration realizes a double-well potential [6], see Fig. 3. How could we change the distance between the two wells?

Answer: The adiabatic potential restricted to the resonance isomagnetic surface thus reads

$$V_m(\mathbf{r}) = m\hbar\Omega_{\rm max}\sqrt{1 - \frac{b'^2x^2}{B_{\rm res}}}$$

with $\hbar\Omega_{\rm max} = g_F\mu_B B_1$. It is minimum at the positions where x is maximum, which defines a double well, with a separation $2r_\perp$ that can be adjusted by changing the rf frequency[2].

3 Spectroscopy of the dressed states

We now go back to the dressed state picture, for the spin + field system, in the rotating wave approximation.

1. Recall the expression of the dressed states $|\tilde{m}\rangle$ and their energy.

[2]See also my lecture notes at the 2013 Les Houches predoc school available on `http://www-lpl.univ-paris13.fr/bec/BEC/Team_Helene.htm`.

Answer: The quantized form of (9) is

$$\hat{H}_{\text{eff}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \omega_0\hat{S}_z + \left[\frac{\Omega_+}{2}\hat{a}\,\hat{S}_+ + \frac{\Omega_+}{2}\hat{a}^\dagger\,\hat{S}_- + \frac{\Omega_-}{2}\,\hat{a}\,\hat{S}_- + \frac{\Omega_-^*}{2}\hat{a}^\dagger\,\hat{S}_+\right].$$

In the RWA, we neglect the two last terms:

$$\hat{H}_{\text{RWA}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \omega_0\hat{S}_z + \frac{\Omega_+}{2}\left(\hat{a}\,\hat{S}_+ + \hat{a}^\dagger\,\hat{S}_-\right).$$

The dressed states are combinations of states of a given manifold

$$\mathcal{E}_N = \{|-S, N\rangle, |-S+1, N+1\rangle, \ldots, |S, N+2S\rangle\}.$$

The coefficients are those of a spin rotation of angle θ around y. We will keep the notation $|m, N\rangle_\theta$ for the dressed states. The associated energies are $m\hbar\sqrt{\delta^2 + |\Omega_+|^2} + (N - S + 1/2)\hbar\omega$.

2. We admit that using a second, weak rf field, it is possible [3] to couple two dressed states $|\tilde{m}\rangle$ and $|\tilde{m}'\rangle$. Explain why we must have $|m - m'| \leq 1$.

Answer: With a weak second rf field, one photon can only couple states which differ by at most 1 in angular momentum. The coupling with involve terms of the form $\hat{a}_{\text{weak}}\hat{S}_{\theta,\pm} + h.c.$ The field can thus couple $|m, N\rangle_\theta$ and $|m', N'\rangle_\theta$ provided that $|m' - m| \leq 1$.

3. Give the possible transition frequencies.

Answer: The possible transitions are

$$\begin{aligned}
&|m, N\rangle \to |m \pm 1, N\rangle && \text{at frequency } |\Omega_+| \\
&|m, N\rangle \to |m \pm 1, N+1\rangle && \text{at frequency } \omega \pm |\Omega_+| \\
&|m, N\rangle \to |m \pm 1, N-1\rangle && \text{at frequency } \omega \mp |\Omega_+|.
\end{aligned}$$

All transitions with frequencies $n\omega \pm |\Omega_+|$ with $n \in \mathbb{N}^*$ are allowed, plus the low frequency $|\Omega_+|$ inside the $\mathcal{E}_N$ manifold.

4. Figure 4 shows rf spectra of rubidium atoms confined in an rf-dressed adiabatic potential. Deduce the Rabi frequency of the rf field from the data.

Answer: Fig. 4 presents the first three transitions at $|\Omega_+|$, $\omega - |\Omega_+|$ and $\omega + |\Omega_+|$. We deduce from the graph that $|\Omega_+| = 2\pi \times 42$ kHz.

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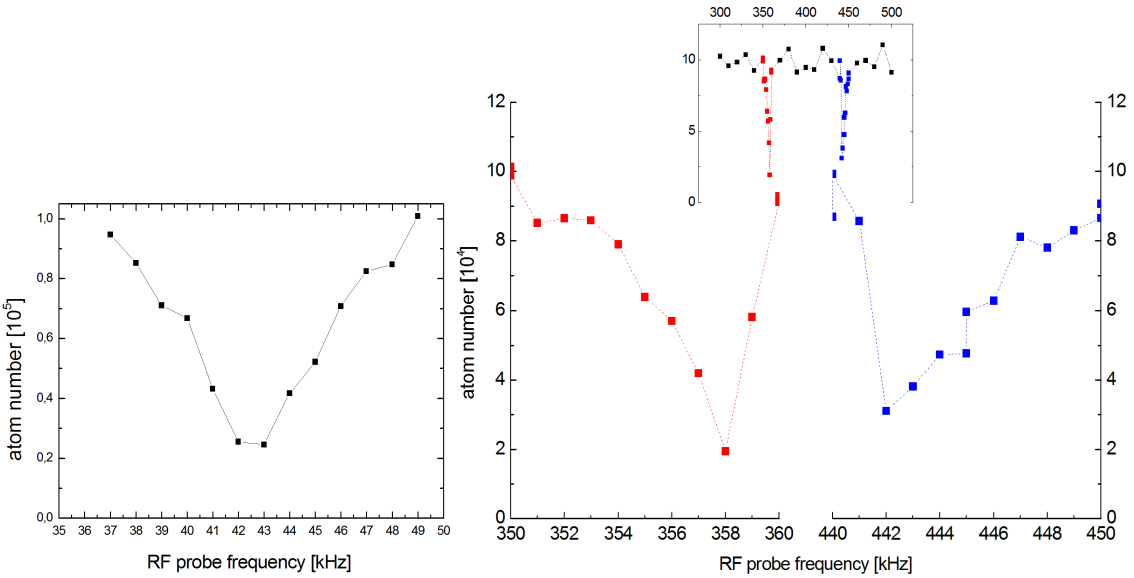

Figure 4: rf-spectroscopy of atoms confined in an adiabatic potential produced by a homogeneous rf field at frequency 400 kHz together with a quadrupole static magnetic field. Spin flips induced by a weak probe produce atom losses, which are recorded as a function of the probe frequency. Losses are observed around 42 kHz, 358 kHz and 442 kHz. Data taken at LPL.

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